Superconformal Couplings of Non-abelian Tensor and Hypermultiplets in Six Dimensions

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M-theory provides extended objects; good progress in the understanding of multiple M2-branes (3D CFT’s BLG & ABJM 08). Finding a “field theory” for multiple M5-branes (6D CFT’s) seems to be much harder:

1.) ¿ Non-abelian Gauge Couplings for Tensor Multiplets ? 
   (2-form Gauge Potentials $B_{\mu\nu}$)

2.) ¿ (2,0) Superconformal Symmetry ?
   (no free parameter/weak coupling parametrization)

- **Ad 1.**: Circumvent no-go theorems (Bekaert, Hennaux, Sevrin) by introducing also gauge fields and 3-form potential $C_{\mu\nu\rho} \Rightarrow “Tensor Hierarchy”$

- **Ad 2.**: Situation similar to M2-branes ( $\mathcal{N} = 8 SO(4)$ BLG “singularity”).
  
  ABJM: $N$ M2-branes on $\mathbb{R}^8/\mathbb{Z}_k \Rightarrow \mathcal{N} = 6$ but $k = 1, 2, \ldots$ weak coupling !!!

6D: Don’t hope for $\mathcal{N} = (2,0)$ “singularity” $\Rightarrow$ Study $\mathcal{N} = (1,0)$ models
Tensor Hierarchy

\[ A^r, B^I, C_r, C_m^{(4)} \quad \ldots \quad 1-, 2-, 3-, 4\text{-form gauge fields} \]

Non-abelian gauge invariance and covariant curvatures
M5-brane zero mode fluctuation (broken susy) $\rightarrow$ Low Energy Theory:

(2,0) Tensor: \( B_{\mu\nu}^{sd} \), \( \Phi^A = 1, \ldots, 5 \), + fermions

(1,0) Tensor: \( B_{\mu\nu}^{sd} \), \( \phi \), \( \chi^i = 1, 2 \), gauge

Vector: \( A_\mu \), \( Y^{ij} \), \( \lambda^i \)

3-Form: \( C_{\mu\nu\rho} \)

(1,0) Hyper: \( q^I \), \( \bar{q}_I = 1, 2 \), + ferm.

(1,0) Tensor: \( B_{\mu\nu}^{sd} \), \( \phi \), \( \chi^i = 1, 2 \), gauge

4-Form: \( C^{(4)} \)

Abelian

\[
\delta B = d\Lambda^{(1)} , \quad \delta A = d\Lambda^{(0)} + \Lambda^{(1)} \quad \Rightarrow \quad \mathcal{F} = F - B
\]

Covariant (abelian $\rightarrow$ invariant)
**Non-abelian:**

**Input 1:** Overcome no-go theorem $\rightarrow$ extend “configuration space”

\[ A^r, \ B^I, \ C^r, \ C_m^{(4)} \ldots \text{ 1-, 2-, 3- and 4-form (in principle up to 6-form)} \]

each form in some representation w/ generalized curvatures:

\[ \mathcal{F}^r, \ \mathcal{H}^I, \ \mathcal{H}^{(4)} \ldots \]

**Input 2:** “Generalized Field Strengths”, via Bianchi Identities

\[ \mathcal{F}^r := F^r_{YM} + h^r_I B^I, \quad D\mathcal{F}^r = h^r_I \mathcal{H}^I \]
Non-abelian:

(de Wit, Nicolai, Samtleben 05/08)

**Input 1:** Overcome no-go theorem → extend “configuration space”

\[ A^r, B^I, C_r, C_m^{(4)} \ldots 1-, 2-, 3- and 4-form (in principle up to 6-form) \]

each form in some representation w/ generalized curvatures:

\[ \mathcal{F}^r, \mathcal{H}^I, \mathcal{H}_r^{(4)} \ldots \]

**Input 2:** “Generalized Field Strengths”, via Bianchi Identities

\[ \mathcal{F}^r := F^r_{YM} + h^r_I B^I, \quad DF^r = h^r_I \mathcal{H}^I \]

\[ \mathcal{H}^I = DB^I + g^{Ir} C_r + d_{rs}^I K^{rs}, \quad D\mathcal{H}^I = d_{rs}^I \mathcal{F}^r \wedge \mathcal{F}^s + g^{Ir} \mathcal{H}_r^{(4)} \]

\[ d + A, \quad dA A, \quad A^3 \]
Non-abelian: \[(\text{de Wit, Nicolai, Samtleben 05/08})\]

**Input 1:** Overcome no-go theorem \(\rightarrow\) extend “configuration space”

\[
A^r, \; B^I, \; C_r, \; C_m^{(4)} \; \ldots \; 1-, \; 2-, \; 3- \; \text{and} \; 4\text{-form} \; (\text{in principle up to} \; 6\text{-form})
\]

each form in some representation w/ generalized curvatures:

\[
\mathcal{F}^r, \; \mathcal{H}^I, \; \mathcal{H}_r^{(4)} \; \ldots
\]

**Input 2:** “Generalized Field Strengths”, via Bianchi Identities

\[
\mathcal{F}^r := F_{YM}^r + h_I^r B^I, \quad D\mathcal{F}^r = h_I^r \mathcal{H}^I
\]

\[
\mathcal{H}^I = D B^I + g^{Ir} C_r + d_{rs}^{I} K^{rs}, \quad D\mathcal{H}^I = d_{rs}^{I} \mathcal{F}^r \wedge \mathcal{F}^s + g^{Ir} \mathcal{H}_r^{(4)}
\]

\[
\mathcal{H}_r^{(4)} = D C_r + k_m^{r} C_m^{(4)} - b_{Irs} (\mathcal{F} B + B^2 + A K + A^4)^I s, \quad D\mathcal{H}_r^{(4)} = -b_{Irs} \mathcal{F}^s \wedge \mathcal{H}^I + k_m^{r} \mathcal{H}_m^{(5)}
\]
Non-abelian:

Input 1: Overcome no-go theorem $\rightarrow$ extend “configuration space”

$$A^r, B^I, C_r, C^{(4)}_m \ldots$$

1-, 2-, 3- and 4-form (in principle up to 6-form)

each form in some representation w/ generalized curvatures:

$$\mathcal{F}^r, \mathcal{H}^I, \mathcal{H}^{(4)}_r \ldots$$

Input 2: “Generalized Field Strengths”, via Bianchi Identities

$$\mathcal{F}^r := F^r_{YM} + h^r_I B^I,$$  \hspace{1cm} $$D\mathcal{F}^r = h^r_I \mathcal{H}^I$$

$$\mathcal{H}^I = DB^I + g^{Ir} C_r + d^I_{rs} K^{rs},$$  \hspace{1cm} $$D\mathcal{H}^I = d^I_{rs} \mathcal{F}^r \wedge \mathcal{F}^s + g^{Ir} \mathcal{H}^{(4)}_r$$

$$\mathcal{H}^{(4)}_r = DC_r + k^r_m C^{(4)}_m - b_{irs} (\mathcal{F} B + B^2 + AK + A^4)^I s,$$  \hspace{1cm} $$D\mathcal{H}^{(4)}_r = -b_{irs} \mathcal{F}^s \wedge \mathcal{H}^I + k^r_m \mathcal{H}^{(5)}_m$$

with $h^r_I, g^{Ir}, d^I_{rs}, b_{irs}, k^r_m$ being invariant tensors of the gauge group

Need to find $\delta A^r, \delta B^I, \delta C_r, \delta C^{(4)}_m$ such that ...
**Condition:** The generalized field strengths must transform **covariantly**:

\[ \delta \mathcal{F}^r = \Lambda^r_s \mathcal{F}^s, \quad \delta \mathcal{H}^I = \Lambda^I_J \mathcal{H}^J, \quad \delta \mathcal{H}^{(4)}_r = \Lambda_r^s \delta \mathcal{H}^{(4)}_s, \text{ e.t.c.} \]
**Condition:** The generalized field strengths must transform covariantly:

\[ \delta F^r = \Lambda^r_s F^s, \quad \delta H^I = \Lambda^I_J H^J, \quad \delta H_r^{(4)} = \Lambda^r_s \delta H_s^{(4)}, \text{ e.t.c.} \]

There is a (unique) solution in terms of \( n = 0, 1, 2, 3 \) - form gauge parameters \( \Lambda^r, \Lambda^I_{(1)}, \Lambda_{(2)r}, \Lambda_{(3)m} \), where the invariant tensors have to satisfy:

\[
(X_r)_s^t = -f_{rs}^t + d_{rs}^I h^t_I, \quad (X_r)_I^J = 2 h^s_I d^J_{rs} - g^J_s b_{Isr}
\]

\[
h^r_I (X_r)_s^t = h^r_I (X_r)_I^J = 0, \quad h^r_I g^I_s = 0, \quad g^Ir k_r^m = 0.
\]

\[
\delta A^r = D \Lambda^r - h^r_I \Lambda^I_{(1)}
\]

\[
\delta B^I = D \Lambda^I_{(1)} - g^Ir \Lambda_{(2)r} + \ldots
\]

\[
\delta C_r = D \Lambda_{(2)r} - k_r \Lambda_{(3)} + \ldots
\]

\[
\Delta g C_m^{(4)} = D \Lambda_{(3)m} + \ldots
\]
Condition: The generalized field strengths must transform covariantly:

\[ \delta F^r = \Lambda^r_s F^s, \quad \delta H^I = \Lambda^I_J H^J, \quad \delta H^{(4)}_r = \Lambda^r_s \delta H^{(4)}_r, \quad \text{e.t.c.} \]

There is a (unique) solution in terms of \( n = 0, 1, 2, 3 \)-form gauge parameters \( \Lambda^r, \Lambda^I_{(1)}, \Lambda_{(2)r}, \Lambda_{(3)m} \), where the invariant tensors have to satisfy:

\[
(X_r)^s_t = -f_{rs}^t + d^I_{rs} h^t_I, \quad (X_r)_I^J = 2 h^s_I d^J_{rs} - g^{Js} b_{Isr}, \quad (k_r^m = 0)
\]

\[
h^r_I (X_r)_s^t = h^r_I (X_r)_I^J = 0, \quad h^r_I g^{Is} = 0, \quad g^{Ir} k_r^m = 0.
\]

\[
\delta A^r = D \Lambda^r - h^r_I \Lambda^I_{(1)}
\]

\[
\delta B^I = D \Lambda^I_{(1)} - g^{Ir} \Lambda_{(2)r} + \ldots
\]

\[
\delta C^r = D \Lambda_{(2)r} - k_r \Lambda_{(3)} + \ldots
\]

\[
\Delta g C^{(4)}_m = D \Lambda_{(3)m} + \ldots
\]

\[ b_{Ir(s} d^I_{pq)} = 0, \quad k_r^m c_{mIJ} = h^s_{[IbJ]rs}, \quad k_r^m c_{m}^t_s = X_{sr}^t - g^{It} b_{Ir}.
\]

\( k_r^m \neq 0 \)
This defines a consistent tensor gauge symmetry, the algebra closes. The invariance of the group tensors \((h_I^T, \ldots)\) gives non-linear algebraic conditions, for which we have classified all solutions.

- **Note: so far no dynamics**
Supersymmetry &
Superconformal Field Equations

On-shell susy determines the dynamics of the tensor multiplet
The task is now to construct susy transformation such that they close into translations and (field dependent) tensor gauge transformations:

\[
[\delta \epsilon_1, \delta \epsilon_2] = \xi^\mu \partial_\mu + \delta (\Lambda^r, \Lambda^{I(r)}, \Lambda^{(2)r}, \Lambda^{(3)m})
\]

- **YM Multiplet:**

  \[
  \delta A^r_\mu = -\bar{\epsilon} \gamma_\mu \lambda^r,
  \]

  \[
  \delta \chi^{ir} = \frac{1}{8} \gamma^{\mu\nu} F^r_{\mu\nu} \epsilon^i - \frac{1}{2} Y^{ijr} \epsilon^j,
  \]

  \[
  \delta Y^{ijr} = -\bar{\epsilon} (i \gamma^\mu D_\mu \lambda^j)^r
  \]

  pure YM \(\Rightarrow\) closes off-shell into transl. + ordinary gauge transf.
The task is now to construct susy transformation such that they close into translations and (field dependent) tensor gauge transformations:

$$\left[\delta \epsilon_1, \delta \epsilon_2\right] = \xi^\mu \partial_\mu + \delta (\Lambda^r, \Lambda^I_{(1)}, \Lambda^I_{(2)}, \Lambda^I_{(3)m})$$

- **YM Multiplet:**

  $$\delta A^r_\mu = -\bar{\epsilon} \gamma_\mu \lambda^r,$$
  $$\delta \chi^i r = \frac{1}{8} \gamma^{\mu \nu} F^r_{\mu \nu} \epsilon^i - \frac{1}{2} Y^{ij} r \epsilon_j + \frac{1}{4} \lambda^r \phi^I \epsilon^i,$$
  $$\delta Y^{ij} r = -\bar{\epsilon} (i \gamma^\mu D_\mu \chi^j) + 2 h^r I \bar{\epsilon} (i \chi) I$$

  still closes off-shell but into tensor gauge transf.
The task is now to construct susy transformation such that they close into translations and (field dependent) tensor gauge transformations:

\[
[\delta \epsilon_1, \delta \epsilon_2] = \xi^\mu \partial_\mu + \delta (\Lambda^r, \Lambda^I_r, \Lambda^r_{(2)}, \Lambda^m_{(3)})
\]

◆ **YM Multiplet:**

\[
\begin{align*}
\delta A^r_\mu &= -\bar{\epsilon} \gamma_\mu \lambda^r, \\
\delta \lambda^{ir} &= \frac{1}{8} \gamma^{\mu\nu} F^r_{\mu\nu} \epsilon^i - \frac{1}{2} Y^{ij r} \epsilon_j + \frac{1}{4} h^r_I \phi^I \epsilon^i, \\
\delta Y^{ij r} &= -\bar{\epsilon} (i \gamma^\mu D_\mu \lambda^{j})^r + 2h^r_I \bar{\epsilon} (i \chi^j)^I
\end{align*}
\]

still closes off-shell but into tensor gauge transf.
The task is now to construct susy transformation such that they close into translations and (field dependent) tensor gauge transformations:

\[
[\delta \epsilon_1, \delta \epsilon_2] = \xi^\mu \partial_\mu + \delta (\Lambda^r, \Lambda^{(1)}_I, \Lambda^{(2)}_r, \Lambda^{(3)}_m)
\]

**YM Multiplet:**

\[
\begin{align*}
\delta A^r_\mu &= -\bar{\epsilon} \gamma_\mu \chi^r, \\
\delta \chi^i_r &= \frac{1}{8} \gamma^{\mu\nu} F^r_{\mu\nu} \epsilon^i - \frac{1}{2} \ Y^{ij}_r \epsilon_j + \frac{1}{4} h^r_I \phi^I \epsilon^i, \\
\delta Y^{ij}_r &= -\bar{\epsilon} (i \gamma^\mu D_\mu \chi^j)^r + 2 h^r_I \bar{\epsilon} (i \chi^j)^I
\end{align*}
\]

still closes off-shell but into tensor gauge transf.

**Tensor Multiplet:**

\[
\begin{align*}
\delta \phi &= \bar{\epsilon} \chi, \\
\delta \chi^i &= \frac{1}{48} \gamma^{\mu\nu\rho} H_{\mu\nu\rho} \epsilon^i + \frac{1}{4} \gamma^\mu \partial_\mu \phi \epsilon^i, \\
\delta B_{\mu\nu} &= -\bar{\epsilon} \gamma_{\mu\nu} \chi
\end{align*}
\]

closes on-shell with \( H^-_{\mu\nu\rho} = 0 \).
Closure of SUSY

The task is now to construct susy transformation such that they close into translations and (field dependent) tensor gauge transformations:

$$\left[ \delta \epsilon_1, \delta \epsilon_2 \right] = \xi^\mu \partial_\mu + \delta (\Lambda^r, \Lambda^I_{(1)}, \Lambda^I_{(2)r}, \Lambda^I_{(3)m})$$

- **YM Multiplet:**
  \[ \begin{align*}
  \delta A^r_\mu &= -\bar{\epsilon} \gamma_\mu \lambda^r, \\
  \delta \chi^{i}^r &= \frac{1}{8} \gamma^{\mu \nu} F^{r}_{\mu \nu} \epsilon^i - \frac{1}{2} Y^{ijr} \epsilon_j + \frac{1}{4} \bar{h}^I_I \phi^I \epsilon^i, \\
  \delta Y^{ijr} &= -\bar{\epsilon}^{(i} \gamma^\mu D^j r \lambda^r) + 2h^r_I \bar{\epsilon}^{(i} \chi^{j)I}
  \end{align*} \]

  still closes off-shell but into tensor gauge transf.

- **Tensor Multiplet:**
  \[ \begin{align*}
  \delta \phi^I &= \bar{\epsilon} \chi^I, \\
  \delta \chi^{i}^I &= \frac{1}{48} \gamma^{\mu \nu \rho} H^{I}_{\mu \nu \rho} \epsilon^i + \frac{1}{4} \gamma^\mu D^I_\mu \phi^I \epsilon + \frac{1}{2} d^I_{rs} \gamma^\mu \chi^{i}^r \epsilon \gamma_\mu \lambda^s, \\
  \Delta B^I_{\mu \nu} &= -\bar{\epsilon} \gamma_{\mu \nu} \chi^I
  \end{align*} \]

  closes on-shell + tensor gauge transf.
Closure of SUSY

The task is now to construct susy transformation such that they close into translations and (field dependent) tensor gauge transformations:

\[
[\delta_1, \delta_2] = \xi^\mu \partial_\mu + \delta(\Lambda^r, \Lambda_I^{(1)}, \Lambda_{(2)r}, \Lambda_{(3)m})
\]

◆ YM Multiplet:

\[
\begin{align*}
\delta A^r_\mu &= -\bar{\epsilon}\gamma_\mu \chi^r, \\
\delta \chi^{i\,r} &= \frac{1}{8} \gamma^{\mu\nu} \mathcal{F}^r_{\mu\nu} \epsilon^i - \frac{1}{2} Y^{ij\,r} \epsilon_j + \frac{1}{4} h^r_I \phi^r_I \epsilon^i, \\
\delta Y^{ij\,r} &= -\bar{\epsilon}^{(i} \gamma^\mu D_\mu \chi^{j)r} + 2h^r_I \bar{\epsilon}^{(i} \chi^{j)I}
\end{align*}
\]

still closes off-shell but into tensor gauge transf.

◆ Tensor Multiplet:

\[
\begin{align*}
\delta \phi^I &= \bar{\epsilon} \chi^I, \\
\delta \chi^{i\,I} &= \frac{1}{48} \gamma^{\mu\nu\rho} \mathcal{H}^I_{\mu\nu\rho} \epsilon^i + \frac{1}{4} \gamma^\mu D_\mu \phi^I \epsilon^i - \frac{1}{2} d^I_{rs} \gamma^\mu \chi^{i\,r} \bar{\epsilon} \gamma_\mu \chi^s, \\
\Delta B^{I}_{\mu\nu} &= -\bar{\epsilon} \gamma_{\mu\nu} \chi^I
\end{align*}
\]

closes on-shell + tensor gauge transf.
The closure of the susy algebra puts the system on-shell (*it's not a bug, it's a feature!*), but does not introduce new conditions on the various constant group tensors. However, susy needs those conditions. ⇒ special Fierz identities!

\[
\Delta C_{\mu\nu\rho r} = -b_{Irs} \bar{\epsilon}_{\mu\nu\rho} \chi^s \phi_I,
\]
\[
\Delta C^{(4)}_{\mu\nu\rho\sigma m} = 2c_{mIJ} \phi^{[I} \bar{\epsilon}_{\mu\nu\rho\sigma} \chi^{J]}
\]
This system of e.o.m. transforms consistently into itself under susy:

- **Novel system of susy non-abelian couplings for multiple (1,0) tensor multiplets in 6D**
- **No dimensionful parameter ⇒ (classical superconformal symmetry).**
- **3-form puts YM on-shell and is dual to vector field**
- **3 & 4 form are non-dynamical ⇒ no new d.o.f.**

The actual model depends on the “choice” of the gauge group/representation and assoc. invariant tensors ⇒ solutions for the various conditions!
With the extended tensor hierarchy (incl. 4-form) one is automatically in the class of models that provide an “auxiliary” action that captures several features of the dynamics. An action also requires a metric $\eta_{IJ}$ for the kin. terms:

$$b_{Ir}(s) d^I_{pq} = 0, \quad h^r_I = \eta_{IJ} g^{Jr}, \quad d_{rs}^I = \frac{1}{2} \eta^{IJ} b_{Jrs}$$

$$L_{\text{tens}} = -\frac{1}{8} (D_\mu \phi^I)^2 - \frac{1}{2} \bar{\chi} I \not\! D \chi^I + \frac{1}{16} b_{Irs} \phi^I L_{\text{YM}}^{rs} - \frac{1}{96} (H_{\mu\nu\rho}^I)^2$$

$$- \frac{1}{48} b_{Irs} \bar{\lambda}^r \not\! D \lambda^s - b_{Irs} \bar{\lambda}^s \left( \frac{1}{4} \not\! F^r - Y^r \right) \chi^I + \frac{1}{8} b_{Irs} g^r_j g^s_k \phi^I \phi^J \phi^K$$

$$+ \frac{1}{2} (b_{jsr} g^s_i - 4 b_{jsr} g^s_j) \phi^I \bar{\lambda}^r \chi^J - \frac{1}{24} b_{Irs} b^I_{uv} \bar{\lambda}^r \gamma^\mu \lambda^u \bar{\lambda}^s \gamma_\mu \lambda^v - \frac{1}{48} L_{\text{top}}$$

- Self duality equ. imposed consistently after deriving 2’nd order equ. (like IIB SUGRA). Problem of a different category.

- VEV of scalar field acts as inverse Yang- Mills coupling, no dimensionful coupling.

- Orthogonality of Stückelberg tensors imply $\eta_{IJ} g^{Ir} g^{Js} = 0 \Rightarrow \text{indefinite metric !}$
topological term:

\[ \int_{\partial M^7} L_{\text{top}} \sim \int_{M^7} (b_I r_s F^r \wedge F^s \wedge H^I - H^I \wedge D H_I) \]

Requirement of gauge invariance might give quantization conditions for the dimensionless coupling constant (see Chern-Simons).

Conformal Symmetry

The action is invariant under the following conformal transformations:

\[ \delta_{\text{conf}} \Phi = L_\xi \Phi + w_\Phi \varpi \Phi \]

w/ \[ \partial_{(\mu \xi \nu)} = w \eta_{\mu \nu} \]

\[ \Phi = (\phi^I, B^I, \chi^I, A^r, Y^{ij}, \lambda^r, C_r) \quad \Rightarrow \quad w_\Phi = (2, 0, 5/2, 0, 2, 3/2, 0) \]

Note: [ Conf, SUSY ] = special SUSY \Rightarrow Superconformal Symmetry
Superconformal Hypers

$\Sigma_{6D}$

$q^A(x)$

$M_{4n}$

Sierra, Townsend (83): 6D susy

de Wit, Rocek, Kleijn, Vandoren (2000/01): superconformal 4D
Fermions in 6D are *symplectic Majorana*-Weyl spinors $\epsilon^i, \psi^a$ e.t.c., while the real scalar fields $q^A$ are coordinates of some target space $(M_{4n}, g)$.

The data for the formulation of 6D $\mathcal{N} = (1,0)$ superconformal $\sigma$-models are then:

\[
\begin{align*}
\delta q^A &= f^A_{\ ia} \epsilon^i \psi^a \\
TM_{4n} &\subseteq Sp(1) \otimes Sp(n) \\
\Rightarrow \quad \text{Hol}(g) &\subseteq Sp(n)
\end{align*}
\]

Thus the target space has to be a hyper Kähler manifold $(M_{4n}, g, \vec{J})$. For conformal symmetry the HK-manifold has to be of special type, namely a HKC:

*Homothetic Killing:* \( \nabla_B \chi^A = \delta^A_B \quad \Rightarrow \quad \mathcal{L}_\chi g_{AB} = 2g_{AB} \quad \text{... Dilatations} \)

The data for the formulation of 6D $\mathcal{N} = (1,0)$ superconformal $\sigma$-models are then:

\[
\begin{align*}
g_{AB} &= \nabla_A \partial_B \chi \quad \Leftrightarrow \quad \chi = \frac{1}{2}g_{AB}\chi^A\chi^B \\
\bar{\omega}_{AB} &= g_{AC} \vec{J}^C_B \\
k^A_D &= 2\chi^A, \quad \vec{k}^A = \frac{1}{2}\vec{J}^A_B\chi^B
\end{align*}
\]

.... metric, HK - potential

.... hyper-Kähler forms

.... *Dilatations, SP(1) R-symmetry*
The superconformal case is closely related to the situation of local susy, i.e. the coupling to supergravity $\Rightarrow$ target space is a $\mathbb{QK}$ manifold $\mathcal{Q}_{4(n-1)}$.

\begin{align*}
\mathcal{M}_{4n} & \quad \text{superconf. quotient (Rocek et al.)} \\
\downarrow & \\
\mathcal{Q}_{4(n-1)} & \quad \text{Swann bundle} \quad 1:1 \text{ correspondence}
\end{align*}

Coupling to supergravity gauges the $Sp(1)$ R-symmetry, which is global in our case. However, we want to consider gauged $\sigma$-models $\Rightarrow$ gauge isometries of HKC while keeping susy and conformal symmetry. These isom. are generated by Killing vectors $X_{(\hat{m})}$:

\begin{align*}
\mathcal{L}_{X_{(\hat{m})}} g_{AB} &= 0 , \\
\mathcal{L}_{X_{(\hat{m})}} \vec{\omega}_{AB} &= 0 , \\
[X_{(\hat{m})}, \vec{k}] &= [X_{(\hat{m})}, k_{D}] = 0
\end{align*}

$\Leftrightarrow$ $G \subseteq \hat{G} = \text{Iso}(\mathcal{Q}_{4(n-1)})$
Superconformal Lagrangian

The last missing geometrical piece for the formulation of a gauged superconformal Lagrangian are the moment maps (Killing potentials) $\vec{\mu}(\hat{m})$:

$$\mathcal{L}_{X(\hat{m})} \tilde{\omega}_{AB} = \partial[A(\tilde{\omega}_B^C X^C_{(\hat{m})})] = 0 \Rightarrow \partial_A \vec{\mu}(\hat{m}) = \tilde{\omega}_{AB} X^B_{(\hat{m})}$$

It remains to choose a gauge group $G \subseteq \text{Iso}(Q_{4(n-1)})$ and gauge the respective isometries:

$$[X_{(m)}, X_{(n)}] = -f_{mn}^p X_{(p)}$$

$$\delta_{\text{Iso}} q^A = \lambda^m X^A_{(m)} \quad \Rightarrow \quad \delta_{\text{G}} q^A(x) := \Lambda^m(x) X^A_{(m)}$$

$$D_\mu q^A = \partial_\mu q^A - A^m_\mu X^A_{(m)}$$
The superconformal Lagrangian is then given by,

\[
\mathcal{L}_{\text{hyp}} = -\frac{1}{2} g_{AB} D_{\mu} q^A D^\mu q^B + \bar{\psi}_a \gamma^\mu (D^a_{\mu} b + \partial_{\mu} q^A \omega_A^a b) \psi^b \\
- \frac{1}{8} W_{abcd} \bar{\psi}^a \gamma^\mu \psi^b \bar{\psi}^c \gamma^\mu \psi^d + 4 \bar{\psi}_a \lambda_i^m f^{ia}_{\ A} X^A_{(m)} + Y_{ij}^m \mu^i_{(m)}
\]

The susy and conformal transformations for the hypermultiplet are

\[
\delta q^A = f^A_{\ ia} \epsilon^i \psi^a, \quad \delta \psi^a = \frac{1}{2} \partial \psi^A E_i f^{ia}_{\ A} - \delta q^A \omega_A^a b \psi^b, \quad \text{(Sierra, Townsend)}
\]

\[
\delta_C q^A = \mathcal{L}_\xi \ q^A + 2 w k_{D}^A(\phi), \quad \delta_C \psi^a = \mathcal{L}_\xi \ \psi^a + \frac{5}{2} w \psi^a - \frac{1}{2} w k_{D}^A \omega_A^a b \psi^b,
\]

with \(\xi\) being the conformal Killing vector field on the world volume \(\Sigma_{6D}\),

\[
\partial_{(\mu} \xi_{\nu)} = w(x) \eta_{\mu \nu}
\]
Vector-Tensor Multiplet

The Lagrangian is supersymmetric under the premise that the vector multiplets \( \mathcal{V}^m = (A^m, \lambda^i{}^m, Y^i{}_{ij}^m) \) obey the standard off-shell susy transformation rules. For a conformal theory one needs also a conformal dynamics for the vector multiplets.

But we have just constructed superconformal models for vector and tensor multiplets. However, the susy transf. of the vector multiplets are modified by \( T^I = (\phi^I, \chi^i{}^I, B^I{}_{\mu\nu}) \):

\[
\begin{align*}
\delta A_r{}^\mu &= -\bar{\epsilon} \gamma_\mu \chi^r, \\
\delta \lambda^i{}^r &= \frac{1}{8} \gamma^{\mu\nu} F_{\mu\nu}^r \epsilon^i - \frac{1}{2} Y^i{}_{ij}{}^r \epsilon^j + \frac{1}{4} h^r{}_{I} \phi^I \epsilon^i, \\
\delta Y^i{}_{ij}{}^r &= -\bar{\epsilon} (i \gamma^\mu D_\mu \chi^j)^r + 2 h^r{}_I \bar{\epsilon} (i \chi^j)^I
\end{align*}
\]

with \( F_{\mu\nu}^r = F^r_{\mu\nu} + h^r{}_I B^I{}_{\mu\nu} \Rightarrow \text{modification triggered by } h^r{}_I \). To provide a superconformal Lagrangian we embed \( \mathcal{V}^m \hookrightarrow \mathcal{V}^r \):

\[
A^m := A^r \theta_r{}^m \quad \text{with: } \quad h^r{}_I \theta_r{}^m \equiv 0 \quad f_{rs}{}^t \theta_t{}^m \equiv \theta_r{}^n \theta_s{}^p f_{np}{}^m
\]

- modification does not affect hypermultiplet
- embedding is homomorph
Target spaces and Gauge groups

It is conjectured that Wolf spaces are all possible compact quaternionic manifolds $\mathbb{Q}_4(n-1)$, of particular interest:

$$\mathbb{H}P^{n-1} = \frac{Sp(n)}{Sp(1) \times Sp(n-1)}$$

The HKC target space for $\mathbb{H}P^{n-1}$ is simply flat $\mathbb{R}^{4n}$ and $\hat{G} = \text{Iso}(\mathbb{Q}_4(n-1)) = Sp(n)$.

$$q^i{}^a := f^i{}^a_A q^A \Rightarrow (q^i{}^a)^* = \varepsilon_{ij} \Omega_{ab} q^j{}^b,$$

$$g_{ij}{}^{ab} = \varepsilon_{ij} \Omega_{ab},$$

$$X^i{}_{(\hat{m})} = u_{(\hat{m})}{}^{ab} q^i{}^b, \quad \mu_{(\hat{m})}^{ij} = (\Omega u_{(\hat{m})})_{ab} q^i{}^a q^j{}^b, \quad u_{(\hat{m})} \ldots \text{sp}(n)$$

**ADE embeddings:**

$$\Omega = \begin{bmatrix} 1 & \mathbb{1} \\ -1 & \mathbb{1} \end{bmatrix} \Rightarrow u = \begin{bmatrix} A & B \\ -B^* & -A^t \end{bmatrix} \text{ with: } A^\dagger = -A, \ B^t = B$$

$$B = 0 : \quad U(n) \hookrightarrow Sp(n)$$

& $A$ real : \quad $So(n) \hookrightarrow Sp(n)$

For $n$ big enough different reps. of the $ADE$ groups can be embedded. The ultimate selection of $ADE$ must come from anomaly cancellation (Blum, Intriligator 97).
Coupling to Non-abelian Tensor Multiplets
Dynamical Conditions

Gauge symmetry of the Tensor Hierarchy ⇒

- Conditions on the group tensors, independent of any dynamics, though important for susy
- Susy e.o.m. of the vector-tensor system allow only for one class of modifications: $k_r^m \mathcal{O}_m$

⇒ For coupling to hypers through gauging, natural to choose the Stückelberg tensor:

$$\theta_r^m = k_r^m \neq 0 \quad \text{and} \quad h^r_l k_r^m = 0 \quad , \quad f_{rs}^t k_t^m = k_r^n k_s^p f_{np}^m$$

This also implies that the four form $\mathcal{C}_4^{(4)}$ is in the adjoint of $G \subseteq \text{Iso}(\mathbb{Q}_{4(n-1)})$:

$$\mathcal{R}_{(m)} = \text{adj}(g)$$
Recall, vector multiplet e.o.m:

\[
\begin{align*}
    b_{Irs} \left( Y_{ij}^s \phi^I - 2 \bar{\lambda}_s^j \chi^I_j \right) &= 0 , \\
    b_{Irs} \left( \mathcal{F}_{\mu\nu}^s \phi^I - 2 \bar{\lambda}_s^I \gamma_{\mu\nu} \chi^I \right) &= \ast \mathcal{H}^{(4)}_{\mu\nu} , \\
    b_{Irs} \left( \phi^I \mathcal{D} \chi^I_i + \frac{1}{2} \mathcal{D} \phi^I \chi^I_i \right) &= b_{Irs} \left[ \frac{1}{4} \mathcal{F}_{\chi}^s \chi_{i}^I + \frac{1}{24} \mathcal{H}^{I} \lambda^s_i - Y_{i\dot{j}}^s \chi^j \right. \\
    &\left. + h^s_J \left( 2 \phi^I \chi^J_i - \frac{1}{2} \phi^J \chi^I_i \right) + \frac{1}{3} d_{uv}^I \gamma^u \chi^I_i \bar{\lambda}^s_{i} \gamma^u \lambda^v \right]
\end{align*}
\]

modulo \( g^{K_r (\ldots)_r} = 0 \) terms

\( \Rightarrow \sim k_r^m \)
Modify vector multiplet e.o.m:

\[ \mathcal{E}(Y) \rightarrow \mathcal{E}(Y) + k_r^m U^i_{jm} \]
\[ \mathcal{E}(\mathcal{F}) \rightarrow \mathcal{E}(\mathcal{F}) + k_r^m V^\mu_{jm} \]
\[ \mathcal{E}(\lambda) \rightarrow \mathcal{E}(\lambda) + k_r^m W'^i_{jm} \]
Modify vector multiplet e.o.m:

\[
\begin{align*}
\mathcal{E}(Y) & \rightarrow \mathcal{E}(Y) + k_r^m \ U^i_j^m, \\
\mathcal{E}(F) & \rightarrow \mathcal{E}(F) + k_r^m \ X^\mu_{\nu}^m, \\
\mathcal{E}(\lambda) & \rightarrow \mathcal{E}(\lambda) + k_r^m \ W^i_m
\end{align*}
\]
Modify vector multiplet e.o.m:

\[
\begin{align*}
E(Y) & \rightarrow E(Y) + k^r_m U^{ij}_m, \\
E(\mathcal{F}) & \rightarrow E(\mathcal{F}) + k^r_m X^{\mu\nu}_m, \\
E(\lambda) & \rightarrow E(\lambda) + k^r_m \mathcal{W}^i_m.
\end{align*}
\]

Closure of the system of e.o.m. under susy leads to the following condition and modifications:

\[
\begin{align*}
\mathcal{D} U_{m\,ij} e^j + 2 \delta \mathcal{W}_{m\,i} &= \varepsilon_{ij} \gamma_\mu j^\mu_m e^j, \\
E(\mathcal{H}^{(5)}) & \rightarrow E(\mathcal{H}^{(5)}) - 2 j^\mu_m, \\
\Delta C^{(4)}_{\mu\nu\rho\sigma\,m} &= 2 c_{m\,IJ} \phi[I\,\bar{\epsilon} \gamma_{\mu\nu\rho\sigma} \chi^J] + \bar{\epsilon} \gamma_{\mu\nu\rho\sigma} \mathcal{W}_m.
\end{align*}
\]

Hypers:

\[
\begin{align*}
U^{ij}_m & \sim \mu^{ij}_m, \quad \mathcal{W}^i_m \sim \psi_a f^{ia}_A X^A_m, \quad j^\mu_m \sim (X_{m\,A} D^\mu q^A - \bar{\psi}_a \gamma^\mu t_m a^b \psi^b).
\end{align*}
\]

Vector, tensor and hypermultiplet are intertwined by the three and four form gauge potential to form a coupled (1,0) on-shell super multiplet.
Adjoint Tensor and Hypermultiplets

For $h^r_I$ being a non-vanishing invariant tensor the representations $\mathcal{R}(r)$, $\mathcal{R}(I)$ have to be reducible:

$$\begin{align*}
\mathcal{R}(r) &= \mathcal{R}(\alpha) \oplus \mathcal{R}(\bar{r}) \quad \ldots \quad \mathcal{V}^r = (\mathcal{V}^\alpha, \mathcal{V}^{\bar{r}}) \\
\mathcal{R}(I) &= \mathcal{R}(a') \oplus \mathcal{R}(\bar{r}) \quad \ldots \quad \mathcal{T}^I = (\mathcal{T}^{a'}, \mathcal{T}^{\bar{r}}) 
\end{align*}$$

$$\Rightarrow h^r_I = \begin{bmatrix} 0 \\ \delta^{\bar{r}} \bar{s} \end{bmatrix}$$

By the orthogonality conditions for the Stückelberg tensors and generators the only non-vanishing components are:

$k^r_m \rightarrow k_{\alpha}^m, \quad g^I r \rightarrow g_{a'}^{\bar{r}}, \quad X_r \rightarrow X_\alpha$

For the rank two tensors to be invariant ($\sim \delta$) one again has to identify representations,

$$\mathcal{R}(\alpha) = \mathcal{R}(m) = \text{adj}(g), \quad \mathcal{R}(a') = \mathcal{R}^c_{(\bar{r})}$$

Thus the coupling to the hypers is rather selective. The only choice left (besides the gauge group $G$) is for the repr. $\mathcal{R}(\bar{r})$ for which we choose the adjoint representation:

$$\mathcal{R}_{\bar{r}} = \text{adj}(g)$$
The resulting field content of the tensor system is then,

\[ \mathcal{R}(r) = \text{adj} \oplus \text{adj} \quad \ldots \quad \mathcal{V}^r = (\mathcal{V}^\alpha, \hat{\mathcal{V}}^\alpha), \quad C_r = (C_\alpha, \tilde{C}_\alpha), \quad C^{(4)}_m = C^{(4)}_\alpha, \]

\[ \mathcal{R}(I) = \text{adj}^c \oplus \text{adj} \quad \ldots \quad \mathcal{T}^I = (\hat{\mathcal{T}}_\alpha, \tilde{\mathcal{T}}^\alpha) \]

\[ \mathcal{R}(hyp) = \mathcal{R}(I) \quad \ldots \quad Q^a = (q^i_a, \psi^a) \rightarrow (\hat{Q}_\alpha, \tilde{Q}^\alpha) \quad \text{i.e.} \quad A = 2 \times \text{adj}(\mathfrak{g}), \quad B = 0 \]

In the adjoint representation (generically) the only invariant rank three tensors are the structure constants \( f_{\alpha\beta\gamma} \) and the d-symbol \( d_{\alpha\beta\gamma} \) of \( \mathfrak{g} = \text{Lie}(G) \). Therefore all the tensors of the hierarchy are expressed through them. The solution to the conditions is:

\[
\begin{align*}
    f_{rs}^{\gamma} &= \begin{bmatrix} f_{\alpha\beta}^{\gamma} & 0 \\ 0 & 0 \end{bmatrix}, \quad f_{rs}^{\tilde{\gamma}} = \frac{1}{2} \begin{bmatrix} 0 & f_{\alpha\beta}^{\gamma} \\ f_{\alpha\beta}^{\gamma} & 0 \end{bmatrix}, \\
    \eta_{IJ} &= \begin{bmatrix} 0 & \delta_{\alpha\beta} \\ \delta_{\alpha\beta} & 0 \end{bmatrix}, \\
    b_{c'}^{\gamma} = 2 d_{c's}^c &= \begin{bmatrix} 0 & -f_{\alpha\beta}^{\gamma} \\ f_{\alpha\beta}^{\gamma} & 0 \end{bmatrix}, \\
    b_{crs} &= 2 d_{c'rs} = 2 d_{\gamma\alpha\beta} \otimes \begin{bmatrix} g & q \\ q & \tau \end{bmatrix}
\end{align*}
\]

To avoid a cubic potential we set in the following \( \tau \equiv 0 \).
The combined dynamic can be summarized by,

\[ \mathcal{L} = \mathcal{L}_{\text{tens}} + \frac{1}{2\lambda} \mathcal{L}_{\text{hyp}} \]

keeping in mind the bookkeeping nature of the tensor Lagrangian (exact only the e.o.m.). Some of the field content is then of the form,

\[ \mathcal{F}^r = \begin{cases} \mathcal{F}^\alpha = F^\alpha \\ \tilde{\mathcal{F}}^\alpha = \tilde{F}^\alpha + \tilde{B}^\alpha \end{cases}, \quad \mathcal{H}^I = \begin{cases} \mathcal{H}_\alpha = D\hat{B}^\alpha + \tilde{C}_\alpha \\ \tilde{\mathcal{H}}^\alpha = D\tilde{\mathcal{F}}^\alpha \end{cases} \]

and one of the interesting couplings is in detail given by,

\[ \mathcal{L}_{\phi\mathcal{F}^2} = -\frac{1}{2} g d_{\alpha\beta\gamma} \tilde{\phi}^\gamma F^\alpha F^\beta - \frac{1}{2} \left[ 2q d_{\alpha\beta\gamma} \tilde{\phi}^\gamma - f_{\alpha\beta\gamma} \hat{\phi}^\gamma \right] F^\alpha \tilde{\mathcal{F}}^\beta \]

where the highlighted coupling is of the form considered by (Blum, Intriligator 97) for anomaly cancellation \( \Rightarrow ADE \).

The new auxiliary field equation is then,

\[ b_{Irs} \left( Y^s_{ij} \phi^I - 2 \tilde{\lambda}^s_{(i} \chi^I_{j)} \right) - \frac{1}{\lambda} k_r^m \mu_m^{ij} = 0 \]
\[ [\tilde{Y}_{ij}, \hat{\phi}] - 2i \{ gY_{ij} + q\tilde{Y}_{ij}, \hat{\phi} \} = P(\lambda, \chi) - \frac{1}{\chi} \mu_{ij}, \]
\[ [Y_{ij}, \hat{\phi}] + 2iq \{ Y_{ij}, \hat{\phi} \} = Q(\lambda, \chi) \]

\( g = q = 0, \ SU(2) : \)

If the \( d \)-symbols, are switched off, \( SU(2) \) has none, one finds constraints:

\[ \text{Tr} [\hat{\phi} ( \frac{1}{\chi} \mu_{ij} - P(\lambda, \chi) ) ] = \text{Tr} [\hat{\phi} Q(\lambda, \chi) ] \overset{!}{=} 0 \]

\[ \mathcal{L}_{Y_{os}} = -\frac{4}{\text{Tr}(\hat{\phi}^2)} \left[ \frac{1}{\chi} \text{Tr} ([\hat{\phi}, \mu^{ij}] \{ \bar{\lambda}_i, \hat{\chi}_j \}) + 2 \text{Tr} (\hat{\phi} \{ \{ \bar{\lambda}^{(i)}, \hat{\chi}^j \}, \{ \bar{\lambda}_i, \hat{\chi}_j \} \} ) \right] \]

No bosonic potential is generated & the conformal point is again “singular”

\( g, q \neq 0, \ U(2) : \)

The abelian factor provides \( d \)-symbols,

\[ T_{\alpha=0,1,2,3} = \left\{ -\frac{i}{2} \mathbb{1}, -\frac{1}{2} \sigma_{\bar{\alpha}} \right\} \Rightarrow d_{(\bar{\alpha} \bar{\beta} 0)} = \delta_{\bar{\alpha} \bar{\beta}}, \ d_{000} = 1 \]

\( \Rightarrow \) No constraints anymore !!!
Conclusions & Outlook

\[ \mathbb{R}^{1,5} \times \mathbb{R} \times \mathbb{C}^2 / \Gamma \]

\[ \phi^I \quad q^A \]
We have constructed 6D (1,0) superconformal models with non-abelian tensor & hypermultiplets. Vector fields (not YM dynamics) are the “glue”.

- non-abelian tensors ⇒ Tensor Hierarchy, non-dynamical 3 & 4-form
- coupling to hypers “selects” type of hierarchy, $G \subseteq \text{Iso}(Q)$
- new multiplet structure, 3,4 form intertwine vectors, tensors & hypers
- perturbatively defined only in the conformally broken phase
- abelian factors are needed to avoid constraints

Questions:

- orbifold target space $(\mathbb{R} \times \mathbb{C}^2 / \Gamma)$, T-duality
- supergravity background $AdS_7/CFT_6$
- tensor hierarchy ⇔ higher gauge theories, non-abelian gerbes
- SD string solutions, BPS states
- Anomaly cancellation
- Reduction to 4/5D ⇔ SYM, Index ...

(R.W. in preparation)
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