Asymptotic Safety in Higher Derivative Gravity

– New Massive Gravity or $D = 3$ –

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Based on
"A Complete Classification of Higher Derivative Gravity in 3D and Criticality in 4D,"
Class. Quant. Grav. 29 (2012) 015002 [arXiv:1109.4458 [hep-th]];

K. Muneyuki and N. Ohta, "Unitarity versus Renormalizability of Higher Derivative Gravity in 3D,"
Phys. Rev. D 85 (2012) 101501 [arXiv:1201.2058];

"Beta Function and Asymptotic Safety in Three-dimensional Higher Derivative Gravity,"
Class. Quant. Grav. 29 (2012) 205012 [arXiv:1205.0476 [hep-th]]

and futher work with R. Percacci.

1 Introduction

Why consider higher derivative gravity?
The quest of quantum theory of gravity is an important subject.
– In 4D, quadratic theory is known to be renormalizable but non-unitary! (Stelle)

Einstein theory is only a low-energy effective theory! If one considers quantum gravity, e.g. string theory, higher-order terms alway appear!

Recent exciting (interesting) developments in 3D gravity:
We have unitary and possibly renormalizable gravity theory.

- **Einstein theory** $\cdots$ **no dynamical degree of freedom**
- **Einstein + CS term** $\cdots$ **massive spin-2 due to higher derivative term**
- **Einstein +** $\alpha R^2 (\alpha > 0)$ $\cdots$ **unitary massive theory**
- $-$**(Einstein)**$+\alpha R^2 + \beta R_{\mu\nu}^2$ $\cdots$ **unitary for** $\alpha = -\frac{3}{8}\beta$ **(new massive gravity).**

Question: What theory is unitary?

2 Unitarity versus Renormalizability

$$S = \frac{1}{\kappa^2} \int d^3x \sqrt{-g} \Big[\sigma R - 2\Lambda_0 + \alpha R^2 + \beta R_{\mu\nu}^2 + \frac{1}{2\mu} \mathcal{L}_{LCS} \Big],$$

This is the higher derivative theory (fourth-order) in three dimensions. (No need to include Riemann squared because no independent Riemann tensor in 3D.)

A complete classification of all unitary and stable theories has recently be done including all possible coupling and maximally symmetric vacua, by looking at the pole residues from the action (off-shell analysis).

N.O., "A Complete Classification of Higher Derivative Gravity in 3D and Criticality in 4D," Class. Quant. Grav. 29 (2012) 015002 [arXiv:1109.4458 [hep-th]].

Table 1: Unitary theories around maximally symmetric spacetimes

α, β	Λ	σ	μ
$\alpha = -\frac{3}{8}\beta, \beta > 0$	negative, $0 > \Lambda > \Lambda_+$	$\sigma = -1$	arbitrary
$\alpha = -\frac{3}{8}\beta, \beta > 0$	positive, $\frac{2}{\beta} > \Lambda > 0$	$\sigma = -1$	arbitrary
$\alpha > 0, \beta = 0$	negative, $0 > \Lambda > -\frac{1}{12\alpha}$	$\sigma = +1$	$\mu = \infty$
$\alpha < 0, \beta = 0$	negative, $\Lambda < -\frac{\sigma}{12\alpha}$, and $\Lambda \leq \frac{\sigma}{12\alpha}$	all	$\mu = \infty$
$\alpha > 0, \beta = 0$	positive, $\frac{1}{20\alpha} \geq \Lambda > 0$	$\sigma = +1$	$\mu = \infty$

The action describes a massless spin-2 graviton, a massive spin-2 and a massive scalar in general.

What is important in this work: It is found that the unitarity of the AdS irrep. is not enough to ensure the unitarity of the field theory, and this can be resolved only by this off-shell analysis.
$\Rightarrow$ New region of unitary theory.

Question: theory can be really both unitary and renormalizable or not?

3 Renormalizability versus Unitarity

The main concern here is whether the theory is really renormalizable or not.
Consider the action

$$S = \frac{1}{\kappa^2}\int d^3x\sqrt{-g}\Big[\sigma R + \alpha R^2 + \beta R_{\mu\nu}^2\Big]$$

Though we can have topological mass term given by the gravitational Chern-Simons term, we do not consider it here for simplicity.
This describes a massless spin-2 graviton (with positive excitation), a massive spin-2 (with negative excitation) and a massive scalar in general. (No need to include Riemann squared because no independent Riemann

tensor in 3D.)

We define the fluctuation around the Minkowski background by

$$\tilde{g}^{\mu\nu} \equiv \sqrt{-g} g^{\mu\nu} = \eta^{\mu\nu} + \kappa h^{\mu\nu}.$$

We find the quadratic term is given by

$$\mathcal{L}_2 = \frac{1}{4} h^{\mu\nu} \Big[P^{(2)}(\beta \Box + \sigma) + P^{(0,s)}\{(8\alpha + 3\beta)\Box - \sigma\} + 2P^{(0,w)}\{(8\alpha + 3\beta)\Box - \sigma\} + \sqrt{2}(P^{(0,sw)} + P^{(0,ws)})\{(8\alpha + 3\beta)\Box - \sigma\} \Big]_{\mu\nu,\rho\sigma} \Box h^{\rho\sigma},$$

where we have defined the projection operators as

$$P^{(2)}_{\mu\nu,\rho\sigma} = \frac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho} - \theta_{\mu\nu}\theta_{\rho\sigma}),$$

$$P^{(1)}_{\mu\nu,\rho\sigma} = \frac{1}{2}(\theta_{\mu\rho}\omega_{\nu\sigma} + \theta_{\mu\sigma}\omega_{\nu\rho} + \theta_{\nu\rho}\omega_{\mu\sigma} + \theta_{\nu\sigma}\omega_{\mu\rho}),$$

$$P^{(0,s)}_{\mu\nu,\rho\sigma} = \frac{1}{2}\theta_{\mu\nu}\theta_{\rho\sigma}, \quad P^{(0,w)}_{\mu\nu,\rho\sigma} = \omega_{\mu\nu}\omega_{\rho\sigma},$$

$$P^{(0,sw)}_{\mu\nu,\rho\sigma} = \frac{1}{\sqrt{2}}\theta_{\mu\nu}\omega_{\rho\sigma}, \quad P^{(0,ws)}_{\mu\nu,\rho\sigma} = \frac{1}{\sqrt{2}}\omega_{\mu\nu}\theta_{\rho\sigma},$$

with

$$\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box}, \qquad \omega_{\mu\nu} = \frac{\partial_\mu \partial_\nu}{\Box}.$$

$P^{(2)}$, $P^{(1)}$, $P^{(0,s)}$ **and** $P^{(0,w)}$ **are the projection operators onto spin 2, 1 and 0 parts, and they satisfy the completeness relation**

$$(P^{(2)} + P^{(1)} + P^{(0,s)} + P^{(0,w)})_{\mu\nu,\rho\sigma} = \frac{1}{2}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\sigma}),$$

on the symmetric second-rank tensors.

After adding the gauge fixing and Faddeev-Popov terms

$$i\delta_B[\bar{c}_\mu(\partial_\nu h^{\mu\nu} - \frac{a}{2}B^\mu)]/\delta\lambda = -B_\mu\partial_\nu h^{\mu\nu} - i\bar{c}_\mu\partial_\nu\mathcal{D}^{\mu\nu}{}_\rho c^\rho + \frac{a}{2}B_\mu B^\mu,$$

(a **is a gauge parameter) the propagator is given by**

$$D_{\mu\nu,\rho\sigma}(k) = \frac{1}{(2\pi)^4}\Big[\frac{P^{(2)}}{k^2(\beta k^2 - \sigma)} + \frac{P^{(0,s)}}{k^2\{(8\alpha + 3\beta)k^2 + \sigma\}} \\ -\frac{a}{2k^2}(2P^{(1)} + 2P^{(0,s)} + P^{(0,w)} - \sqrt{2}(P^{(0,sw)} + P^{(0,ws)})\Big]_{\mu\nu,\rho\sigma}.$$

The most important property:

It damps as k^{-4} **for large momentum.**

$\Rightarrow$ **The theory becomes power-counting super-renormalizable.**

Follow the usual power counting:

Notations:

V_2**: the number of graviton vertices with two derivatives from** R **term,**

V_4: the number of graviton vertices with four derivatives from R^2 term,
V_c: the number of ghost-anti-ghost-graviton vertices with two derivatives,
V_K: the number of K-graviton-ghost vertices,
V_L: the number of L-ghost-ghost vertices, **(L and K are vertices introduced to make Slavnov-Taylor identities closed.)**
I_h: the number of internal-graviton propagators,
I_c: the number of internal-ghost propagators,
E_h: the number of external gravitons,
E_c: the number of external ghosts.

The degree of divergence of an arbitrary 1PI diagram is given by

$$D^{(1PI)} = 3 - (I_h - V_4) - V_2 - \frac{3}{2}V_K - V_L - \frac{5}{2}(E_c + E_{\bar{c}}).$$

$I_h - V_4 \geq 0$ for 1PI diagrams.

Consequently we find that the possible divergences are restricted; those with external ghost and anti-ghost have $D^{(1\text{PI})} \leq -2$, those with the external K and ghost $D^{(1\text{PI})} \leq -1$, and those with L and two ghosts $D^{(1\text{PI})} \leq -3$.

This allows only the counterterms of the Einstein and the cosmological terms. Hence the theory is super-renormalizable.

However, there are two important exceptional cases:

$8\alpha + 3\beta = 0$ **and** $\beta = 0 \Rightarrow$ **makes the behavior of the propagator** $\sim k^{-2}$
The former (latter) case corresponds to the new massive ($f(R)$) gravity. Although this theory is found to be unitary either for $8\alpha + 3\beta = 0$ **or** $\beta = 0$**, the renormalizability fails precisely in these cases and the theory is not renormalizable. It may appear that the scalar modes decouples for** $8\alpha + 3\beta = 0$ **but the interaction breaks the decoupling.**
"Unitarity versus Renormalizability of Higher Derivative Gravity in 3D," K. Muneyuki and N. Ohta, Phys. Rev. D 85 (2012) 101501 [arXiv:1201.2058].

In this situation, the only possible way to make sense of the quantum effects in gravity seems to be the asymptotic safety.

What is the asymptotic safety?

A non-renormalizable theory $\Rightarrow$ effectively renormalizable by a rearrangement of the perturbation series or addition of higher derivative terms.

Terms of finite order in the perturbation series then contain what appear to be unphysical singularities. Such unphysical singularities may be almost certainly avoided if the couplings approach a fixed point in the ultraviolet energy. This is the asymptotic safety (Weinberg).

The asymptotic safety is a wider notion than the renormalizability (includes renormalizable theories).

4 Wilsonian method for renormalization group

Wilsonian RG:
Effective action describing physical phenomenon at a momentum scale k = the result of integrating out all fluctuations of the fields with momenta larger than k.
k: the lower limit of the functional integration (the infrared cutoff). The dependence of the effective action on k gives the Wilsonian RG flow.

At the one loop, Γ_k is given by

$$\Gamma_k^{(1)} = S + \frac{1}{2}\mathbf{Tr}\log(S^{(2)} + R_k),$$

where $S^{(2)}$ denotes the second variation of the bare action. We then obtain

$$k\frac{d\Gamma_k^{(1)}}{dk} = \frac{1}{2}\mathbf{Tr}(S^{(2)} + R_k)^{-1} k\frac{dR_k}{dk}.$$

R_k is introduced as the cutt-off. The factor $k\frac{dR_k}{dk}$ goes to zero for $z > k^2$.
One can obtain the one-loop beta functions from this functional equation.

We apply this method to our theory on maximally symmetric spaces.

Suitably fix the gauge parameters and then compute beta functions.

The beta functions were computed for maximally symmetric spacetime, but then those for α and β cannot be computed separately.

"Beta Function and Asymptotic Safety in Three-dimensional Higher Derivative Gravity," Class. Quant. Grav. 29 (2012) 205012 [arXiv:1205.0476 [hep-th]].

$\Rightarrow$ Assuming the existence of fixed points for α and β, non-trivial fixed points were found for Λ and G.

4.1 $\sigma = +1$

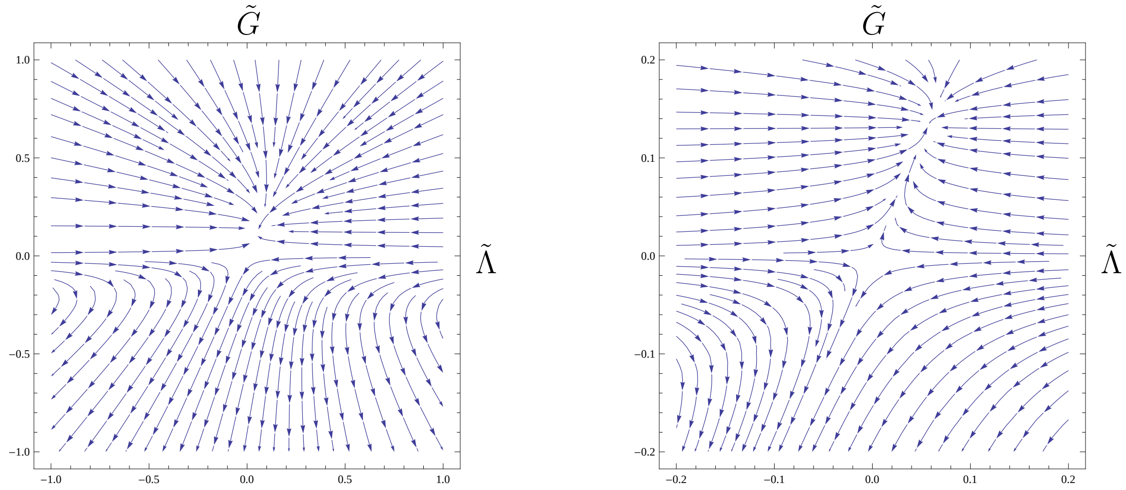
The RE equations have two fixed points:

The Gaussian fixed point $\tilde{G} = \tilde{\Lambda} = 0$, attractive in the $\tilde{\Lambda}$ direction and repulsive in the $\tilde{G}$ direction.

Another nontrivial one

$$\tilde{G}_* = -\frac{1}{B}, \quad \tilde{\Lambda}_* = \frac{A_0}{A_1 + 6B}, \quad \text{for some constants } A_0, A_1, B.$$

The absolute value of $\tilde{G}_*$ is very small for typical values of $\alpha = -1$ and



(a) (b)

Figure 1: The RG flow (a) for $\alpha = -1, \beta = 3$ and $b = 0$. (b) Magnified one near origin.

$\beta = 3$ (and $b = 0$), so that this is certainly within the perturbative domain.

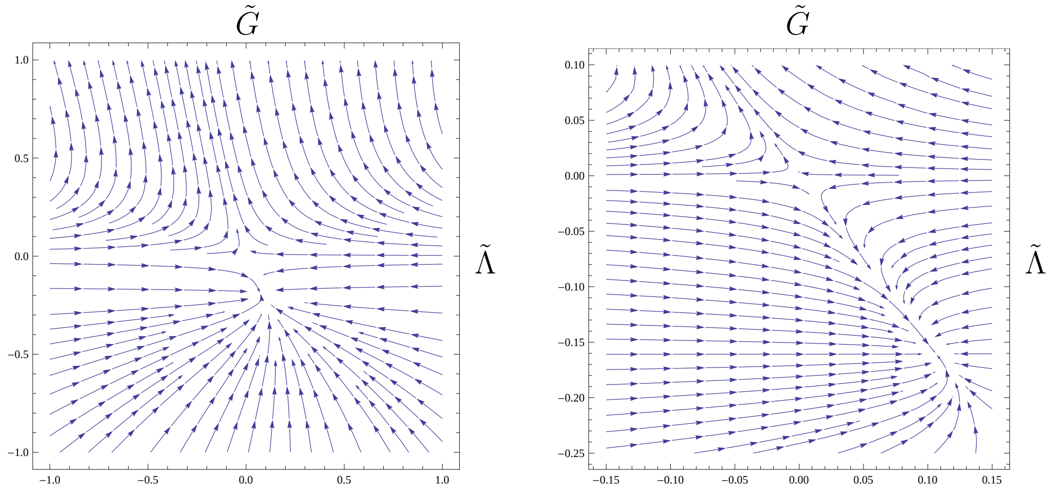
The fixed point value of the cosmological constant is gauge-invariant, positive and small!

4.2 $\sigma = -1$

The Gaussian fixed point, attractive in the $\tilde{\Lambda}$ direction and repulsive in the $\tilde{G}$ direction.

Another nontrivial fixed point is given by

$$\tilde{G}_* = \frac{1}{B}, \quad \tilde{\Lambda}_* = \frac{A_0}{-A_1 + 6B}.$$



(a) (a)

Figure 2: The flow is shown for $\alpha = -1, \beta = 3$ and $b = 0$. (b) is a magnified diagram.

The absolute value of $\tilde{G}_*$ is very small and negative, resulting in positive Einstein term!

The fixed point value of the cosmological constant is again gauge-invariant, positive and small!

4.3 General background

Work with Roberto.

We derive the beta functions for general background and dimensions.

$$\beta_{\tilde\alpha} = \tilde\alpha - \frac{1}{960\pi^2}\left[\left(1+\frac{47\tilde\alpha}{\tilde\beta}+\frac{64\tilde\alpha^2}{\tilde\beta^2}-\frac{128\tilde\alpha}{2\tilde\alpha+\tilde\beta}\right)\sqrt{-\frac{2\tilde\beta}{\tilde\alpha}}\right.$$
$$\left.+8\left(129+\frac{46\tilde\alpha}{\tilde\beta}-\frac{114\tilde\alpha^2}{\tilde\beta^2}+\frac{32\tilde\alpha}{2\tilde\alpha+\tilde\beta}+\tilde\alpha\frac{131\tilde\alpha+49\tilde\beta}{(8\tilde\alpha+3\tilde\beta)^2}\right)\right],$$

$$\beta_{\tilde\beta} = \tilde\beta + \frac{1}{1440\pi^2}\left[-3\left(1-\frac{113\tilde\alpha}{\tilde\beta}+\frac{64\tilde\alpha^2}{\tilde\beta^2}+\frac{192\tilde\alpha}{2\tilde\alpha+\tilde\beta}\right)\sqrt{-\frac{2\tilde\beta}{\tilde\alpha}}\right.$$
$$\left.+8\left(413-\frac{93\tilde\alpha}{\tilde\beta}+\frac{72\tilde\alpha^2}{\tilde\beta^2}+\frac{144\tilde\alpha}{2\tilde\alpha+\tilde\beta}+\tilde\alpha\frac{652\tilde\alpha+243\tilde\beta}{(8\tilde\alpha+3\tilde\beta)^2}\right)\right].$$

These form a closed system of equations with a fixed point at $\tilde\alpha = 0.04962$, $\tilde\beta = -0.1381$.

It would be more appropriate to consider the beta functions for $\lambda = 1/\beta$ and $\omega = -2\alpha/\beta$. We find that they are given by

$$\beta_\lambda = -\lambda - \frac{\lambda^2}{480\pi^2\sqrt{\omega}(\omega-1)}\Big[2 - 81\omega - 81\omega^2 - 32\omega^3 + \frac{8\sqrt{\omega}(-1239 + 4660\omega - 5720\omega^2 + 2219\omega^3 + 8\omega^4 + 96\omega^5)}{(4\omega-3)^2}\Big],$$

$$\beta_\omega = -\frac{\lambda}{480\pi^2\sqrt{\omega}(\omega-1)}\Big[(1+\omega)(2 + 79\omega - 81\omega^2 - 32\omega^3) + \frac{2\sqrt{\omega}(4644 - 23468\omega + 43055\omega^2 - 32209\omega^3 + 5788\omega^4 + 1856\omega^5 + 384\omega^6)}{(4\omega-3)^2}\Big].$$

This indicates that we have fixed points at $\lambda = 0$, and $(\lambda, \omega) = (-7.240, 0.7187)$.
The beta functions of $\tilde{\Lambda}$ and $\tilde{G}$ are:

$$\begin{aligned} \beta_{\tilde{\Lambda}} &= -2\tilde{\Lambda} - p(\tilde{\alpha},\tilde{\beta})\tilde{\Lambda} - q(\tilde{\alpha},\tilde{\beta})\tilde{G}\tilde{\Lambda} + r(\tilde{\alpha},\tilde{\beta})\tilde{G} + s(\tilde{\alpha},\tilde{\beta}) + \frac{t(\tilde{\alpha},\tilde{\beta})}{\tilde{G}} , \\ \beta_{\tilde{G}} &= \tilde{G} + u(\tilde{\alpha},\tilde{\beta})\tilde{G} - q(\tilde{\alpha},\tilde{\beta})\tilde{G}^2 , \end{aligned}$$

where

$$p(\tilde{\alpha},\tilde{\beta}) = \frac{-2304\tilde{\alpha}^3 + 1120\tilde{\alpha}^2\tilde{\beta} + 1778\tilde{\alpha}\tilde{\beta}^2 + 387\tilde{\beta}^3}{48\pi^2(8\tilde{\alpha}+3\tilde{\beta})^2\tilde{\beta}^2},$$

$$q(\tilde{\alpha},\tilde{\beta}) = \frac{1}{3\pi}\left[37 + \frac{\tilde{\beta}}{8\tilde{\alpha}+3\tilde{\beta}} - \frac{16\tilde{\alpha}}{\tilde{\beta}} + \sqrt{-\frac{2\tilde{\beta}}{\tilde{\alpha}}} + 4\left(1 - \frac{8\tilde{\alpha}}{\tilde{\beta}}\right)\sqrt{\frac{-2\tilde{\alpha}}{\tilde{\beta}}}\right],$$

$$r(\tilde{\alpha},\tilde{\beta}) = \frac{1}{3\pi^2}\left[24 - \frac{1}{2}\left(\frac{-2\tilde{\beta}}{\tilde{\alpha}}\right)^{3/2} - 8\left(\frac{-2\tilde{\alpha}}{\tilde{\beta}}\right)^{3/2}\right],$$

$$s(\tilde{\alpha},\tilde{\beta}) = \frac{16\tilde{\alpha}+5\tilde{\beta}}{8\pi^2(8\tilde{\alpha}+3\tilde{\beta})\tilde{\beta}},$$

$$t(\tilde{\alpha},\tilde{\beta}) = 3\frac{128\tilde{\alpha}^2+96\tilde{\alpha}\tilde{\beta}+19\tilde{\beta}^2}{1024\pi^3(8\tilde{\alpha}+3\tilde{\beta})^2\tilde{\beta}^2},$$

$$u(\tilde{\alpha},\tilde{\beta}) = \frac{2304\tilde{\alpha}^3+2912\tilde{\alpha}^2\tilde{\beta}+1174\tilde{\alpha}\tilde{\beta}^2+153\tilde{\beta}^3}{48\pi^2(8\tilde{\alpha}+3\tilde{\beta})^2\tilde{\beta}^2}.$$

If we choose $\tilde{\alpha}$ and $\tilde{\beta}$ at their fixed point, one finds a fixed point at $\tilde{\Lambda} = 0.5355$ and $\tilde{G} = 0.1758$.

5 Discussions

We have summarized the unitarity, stability and renormalizability of 3D higher derivative gravities.

It is quite surprising and/or disappointing that the unitarity and renormalizability are incompatible with each other.

The only way to make sense of ultraviolet behaviors of the quantum effects in gravity is the asymptotic safety. We have studied if this is realized in our higher derivative theories.

Results:

1. **There are Gaussian and nontrivial fixed points for both signs of the Einstein term. The theory is asymptotic safe.**
2. **The fixed points of the gravitational constant is positive and small. The perturbative results are valid.**
3. **The fixed point values of the cosmological constants are gauge-independent, positive and small. ⇒ Physical implication?**

Possible future directions:

extending the analysis to more general theory. ⇒ Supergravity, Massive gravity, Mukohyama's theory, etc. More fun!!