On the completion of the one-loop $R^4$ coupling in type II string theory

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JTL and R. Minasian, arXiv:1304.3137
The low energy limit of string theory

- In the classical limit, string theory reduces to an effective theory of gravity + supersymmetry
  - For the type IIA string
    \[ \mathcal{L} = e^{-2\phi} \left[ R \ast 1 + 4d\phi \wedge \ast d\phi - \frac{1}{2} H \wedge \ast H \right] \]
    \[ - \frac{1}{2} F_2 \wedge \ast F_2 - \frac{1}{2} F_4 \wedge \ast F_4 - \frac{1}{2} B_2 \wedge F_4 \wedge F_4 \]
  - Includes both NSNS and RR fields

- In many applications, this is all we need
  - Classical supergravity solutions
  - Supersymmetric backgrounds, string compactification
  - AdS/CFT in the large \( N \), large \( \lambda \) limit
Beyond the low energy limit

- Two categories of stringy corrections
  1. String loop corrections with $g_s \sim e^\phi$
  2. String $\alpha'$ corrections (higher derivative corrections)

- Distinguishes string theory from supergravity

$$\mathcal{L} = e^{-2\phi}\mathcal{L}_0 + \mathcal{L}_1 + e^{2\phi}\mathcal{L}_2 + \cdots$$

$$\mathcal{L}_g \sim \sum_{n \geq 1} (\alpha')^{n-1} (2n \text{ derivative terms})$$

- In AdS/CFT

$$g_s \rightarrow \frac{\lambda}{N}, \quad \alpha' \rightarrow \frac{1}{\sqrt{\lambda}}$$

so stringy corrections give us information about finite coupling and finite $N$ effects
Higher curvature corrections in string theory

▶ How do we determine the stringy corrections?
  - Reconstruct an effective action from string scattering amplitude calculations
  - For $\alpha'$ corrections, perform sigma-model loop calculations
  - Construct supersymmetric higher derivative invariants
  - Anomalies, consistency, symmetries, etc.

▶ For the type II string, the first correction arises at the level of the four-point function

$$\mathcal{A}_4 \sim h^{(1)} h^{(2)} h^{(3)} h^{(4)}$$

- But the result must be gauge invariant

$$h \rightarrow k_{[\mu} k^{[\rho} h_{\nu]} \sigma] \sim R_{\mu\nu}^{\rho\sigma}$$

- So the four-point function gives an $\alpha'^3 R^4$ correction to the effective action (at tree-level and one-loop)
For the type II string at $O(\alpha'^3)$

$$e^{-1}\mathcal{L} \sim e^{-2\phi}(t_8 t_8 R^4 + \frac{1}{8} \epsilon_{10} \epsilon_{10} R^4) + (t_8 t_8 R^4 \mp \frac{1}{8} \epsilon_{10} \epsilon_{10} R^4) + (t_8 \epsilon_{10} \pm \epsilon_{10} t_8) B R^4$$

- At one loop in the RNS formalism

  - even spin structure $\rightarrow t_8$
  - odd spin structure $\rightarrow \epsilon_{10}$

The CP-odd term vanishes for IIB
- For IIA it has the familiar expression

$$B \wedge X_8(R) = \frac{1}{192(2\pi)^4} B \wedge [\text{Tr } R^4 - \frac{1}{4} (\text{Tr } R^2)^2]$$

- Related to the five-brane anomaly
Completing the $R^4$ couplings

- The $R^4$ terms as written are necessarily incomplete
  - The full supergravity multiplet must show up

  Part of a higher derivative supersymmetry invariant

  - What about T-duality?

  \[
g_{\mu\nu} \leftrightarrow B_{\mu\nu}
\]

- We focus on the NSNS sector only: \((g_{\mu\nu}, B_{\mu\nu}, \phi)\)
  - Then, at the two-derivative level

  \[
  R \sim \nabla H \sim H^2 \sim \nabla \nabla \phi \sim (\nabla \phi)^2
  \]

- so, for example

  \[
  R^4 \sim R^2(\nabla H)^2 \sim H^2 R^3 \sim \cdots \sim H^8
  \]
Introduce the connection with torsion

- How we compute the closed string four-point function

\[ V^{(-1,-1)}(k, \theta) = \theta_{\mu\nu} \delta(\gamma) \delta(\tilde{\gamma}) \psi^\mu \tilde{\psi}^\nu e^{ik \cdot X} \]

where the polarization tensor is

\[ \theta_{\mu\nu} = h_{\mu\nu} + b_{\mu\nu} + \frac{1}{2}(\eta_{\mu\nu} - k_\mu k_\nu - k_\mu \bar{k}_\nu) \phi \]

- Captures the scattering of any combination of NSNS fields
- Instead of \( k_\mu k^{[\rho h_\nu]} \sigma \sim R_{\mu\nu}{}^{\rho \sigma} \), we have

\[ k_\mu k^{[\rho \theta_\nu]} \sigma \sim R_{\mu\nu}{}^{\rho \sigma} + e^{-\phi/2} \nabla_{[\mu} H_{\nu]}{}^{\rho \sigma} - \delta_{[\mu}{}^{[\rho} \nabla_{\nu]} \nabla^{\sigma]} \phi \]

  - Simpler when Weyl scaled to the string frame

\[ \rightarrow \quad R_{\mu\nu}{}^{\rho \sigma} + \nabla_{[\mu} H_{\nu]}{}^{\rho \sigma} \]

- This is the linearized curvature computed from a connection with torsion

\[ R(\Omega_+) \quad \text{where} \quad \Omega_\pm = \omega \pm \frac{1}{2} \mathcal{H} \]
What we expect

- At the linearized level $R \rightarrow R(\Omega_+)$ is the correct thing to do
  - The one loop couplings receive contributions from the difference spin structure sectors

<table>
<thead>
<tr>
<th></th>
<th>Without $B$</th>
<th>With $B$</th>
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<tbody>
<tr>
<td><strong>CP-even</strong></td>
<td></td>
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<tr>
<td>e-e</td>
<td>$t_8t_8R^4$</td>
<td>$t_8t_8R(\Omega_+)^4$</td>
</tr>
<tr>
<td>o-o</td>
<td>$\epsilon_{10}\epsilon_{10}R^4$</td>
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</tr>
<tr>
<td><strong>CP-odd</strong></td>
<td></td>
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</tr>
<tr>
<td>e-o &amp; o-e</td>
<td>$B \wedge X_8(\omega)$</td>
<td>$\frac{1}{2}B \wedge [X_8(\Omega_+) + X_8(\Omega_-)]$</td>
</tr>
</tbody>
</table>


- But is this complete (in the NSNS sector) at the non-linear level?
But wait . . . there’s more

- Use of the connection with torsion is compatible with T-duality
- However, there are hints that this is incomplete
  - Different symmetry properties

\[ R_{\mu\nu\rho\sigma} : \quad vs. \quad \nabla_\mu H_{\nu\rho\sigma} : \]

- String three-point function computed for type II on K3

\[
\begin{align*}
\text{e-e:} & \quad R_{\mu\nu\rho\sigma}^2 + \frac{1}{2} \nabla_\mu H_{\nu}^{\ ab} \nabla_\nu H_{\mu}^{\ ab} \rightarrow t_4 t_4 R(\Omega_+)^2 \\
\text{o-o:} & \quad R_{\mu\nu\rho\sigma}^2 + \frac{1}{2} \nabla_\mu H_{\nu}^{\ ab} \nabla_\nu H_{\mu}^{\ ab} \rightarrow -\frac{1}{8} \epsilon_6 \epsilon_6 R(\Omega_+)^2 + H_{\mu\rho}^a H_{\nu\sigma}^a R^{\mu\nu\rho\sigma}
\end{align*}
\]


- We conjecture

\[ R \rightarrow R(\Omega_+) \text{ complete except in the o-o sector} \]
Evidence supporting this conjecture

- Beyond the linearized Riemann contribution

\[
R(\Omega_{\pm})_{\mu\nu}^{\alpha\beta} = R_{\mu\nu}^{\alpha\beta} \pm \nabla_{[\mu} H_{\nu]}^{\alpha\beta} + \frac{1}{2} H_{[\mu}^{\alpha\gamma} H_{\nu]}^{\gamma\beta}
\]

so

\[
R(\Omega_{+})^4 \sim R^4 + \cdots + H^8 \quad \text{(automatically even in } H)\]

- Conjecture: this is all that is needed in the e-e sector and the CP-odd sector

- Some evidence
  - T-duality in the CP-odd sector
  - Heterotic/IIA duality
  - String four-point function for type II on K3 in the o-o sector
A note on the CP-odd sector

- The $B \wedge X_8$ term is related to anomalies, and is highly constrained.
- Consider the shift $B \wedge X_8 \rightarrow B \wedge \tilde{X}_8 = B \wedge (X_8 + Y)$
  - Invariance under $B$ field gauge transformations: $dY = 0$
  - NS5-brane anomaly cancellation: $Y$ should be exact
- The replacement $\omega \rightarrow \omega + \frac{1}{2} \mathcal{H}$ automatically satisfies both conditions.
- For example, take

$$8\pi^2 X_4 = \text{Tr} R(\Omega_+) \wedge R(\Omega_+)$$

$$= \text{Tr} R \wedge R + d[\text{Tr} (R \wedge \mathcal{H} + \frac{1}{4} \mathcal{H} \wedge d\mathcal{H} + \frac{1}{12} \mathcal{H}^3)]$$
T-duality in the CP-odd sector

▶ Consider a circle reduction at constant radius

\[ ds^2 = \eta_{\alpha\beta} e^\alpha e^\beta + e^9 e^9, \quad de^9 = T \]

\[ B_2 = b_2 + b_1 \wedge e^9, \quad H_3 = h_3 + \tilde{T} \wedge e^9, \quad db_1 = \tilde{T} \]

▶ Let \( X_4 = \tilde{X}_4 + \tilde{X}_3 \wedge e^9 \)

▶ Then

\[ \int_{\mathcal{M}_6} B \wedge X_4(\Omega_+) = \int_{\mathcal{M}_5} b_1 \wedge (\tilde{X}_4 - \tilde{X}_2 \wedge T) - h_3 \wedge \tilde{X}_2 \]

where

\[-8\pi^2 (\tilde{X}_4 - \tilde{X}_2 \wedge T) = R^{\alpha\beta}(\omega_+) \wedge R^{\alpha\beta}(\omega_+) \]

\[ -\frac{1}{2} R^{\alpha\beta}(\omega_+) \wedge T_{+\gamma} e^\gamma \wedge T_{+\delta} e^\delta \]

\[ + \frac{1}{8} D_\alpha(\omega_-) T_+ \wedge D^\alpha(\omega_-) T_+ \]

▶ This expression is T-duality invariant
Heterotic/IIA duality

- Consider the duality between Heterotic on $T^4$ and IIA on $K3$
- The tree-level $R^2$ correction is known

\[
e^{-1} \mathcal{L}_{d=6}^{\text{het}} = e^{-2\phi} [R + 4\partial\phi^2 - \frac{1}{12} H_{\mu\nu\rho}^2 + \frac{1}{8}\alpha' R_{\mu\nu\rho\sigma}(\Omega_+)^2 + \cdots ]
\]

\[
dH = \frac{1}{4}\alpha' \text{Tr} R(\Omega_+) \wedge R(\Omega_+)
\]

- So all we need to do is dualize

\[
e^{-2\phi} H^{\text{het}} = *H^{\text{IIA}}, \quad e^{-2\phi} g_{\mu\nu}^{\text{het}} = g_{\mu\nu}^{\text{IIA}}, \quad \phi = -\phi^{\text{IIA}}
\]

(We find that no correction is needed at $O(\alpha')$)

- Note that the connection with torsion does not map directly

\[
\Omega_+^{\text{het}} \not\to \Omega_+^{\text{IIA}}
\]
The resulting IIA Lagrangian

In the IIA frame \((dH = 0)\), we find

\[
e^{-1} \mathcal{L}_{d=6}^{\text{IIA}} = e^{-2\varphi} [R + 4\partial\varphi^2 - \frac{1}{12} H_{\mu\nu\rho}] \\
+ \frac{1}{16} \alpha' [t_4 t_4 R(\Omega_+)^2 - \frac{1}{8} \epsilon_6 \epsilon_6 (R(\Omega_+)^2 + \frac{4}{3} H^2 R(\Omega_+) + \frac{1}{9} H^4)] \\
- \frac{1}{8} \alpha' B \wedge [\text{Tr } R(\Omega_+) \wedge R(\Omega_+) + \text{Tr } R(\Omega_-) \wedge R(\Omega_-)]
\]

where

\[
t_4 t_4 R(\Omega_+)^2 = R_{\mu\nu\rho\sigma}(\Omega_+) R^{\mu\nu\rho\sigma}(\Omega_+)
\]
\[
\epsilon_6 \epsilon_6 R(\Omega_+)^2 = \epsilon_{\alpha\beta\mu_1 \cdots \mu_4} \epsilon^{\alpha\beta\nu_1 \cdots \nu_4} R^{\mu_1\mu_2\nu_1\nu_2}(\Omega_+) R^{\mu_3\mu_4\nu_3\nu_4}(\Omega_+) \\
- 8 (R_{\mu\nu\rho\sigma}(\Omega_+) R^{\rho\sigma\mu\nu}(\Omega_+) - 4 R_{\mu\nu}(\Omega_+) R^{\nu\mu}(\Omega_+) + R(\Omega_+)^2)
\]
\[
\epsilon\epsilon H^2 R(\Omega_+) = \epsilon_{\alpha\mu_0 \cdots \mu_4} \epsilon^{\alpha\nu_0 \cdots \nu_4} H^{\mu_1\mu_2 \nu_1\nu_2} H^{\mu_3\mu_4 \nu_3\nu_4}(\Omega_+) \\
\epsilon\epsilon H^4 = \epsilon_{\alpha\beta\mu_1 \cdots \mu_4} \epsilon^{\alpha\beta\nu_1 \cdots \nu_4} H^{\mu_1\mu_2\rho} H^{\nu_1\nu_2\rho} H^{\mu_3\mu_4\sigma} H^{\nu_3\nu_4\sigma}
\]

Only the o-o contribution receives additional corrections.
The string one-loop four-point function

- In the o-o sector, we take one vertex operator in the \((-1, -1)\) picture and three in the \(0, 0\) picture

\[
\mathcal{A} \sim \theta_{\mu_1\nu_1}^{(1)} \theta_{\mu_2\nu_2}^{(2)} \theta_{\mu_3\nu_3}^{(3)} \theta_{\mu_4\nu_4}^{(4)} \\
\times \left\langle \psi \cdot \partial X(0) \psi^{\mu_1} \prod_{i=2}^{4} \left( i\partial X^{\mu_i} + \frac{1}{2} \alpha' k_i \cdot \psi \psi^{\mu_i} \right) \prod_{i=1}^{4} e^{ik_i \cdot X} \right\rangle
\]

- The computation is simplified because we must soak up six fermion zero modes on each side of the string
  - Multiply out the vertex operators, and keep the six-fermion and eight-fermion terms on each side
  - We focus on the kinematical factors, and do not explicitly perform the modular integrals

- We find

\[
e^{-1} \mathcal{L}_{o-o} \sim \epsilon_6 \epsilon_6 (R(\Omega_+)^2 + \frac{4}{3} H^2 R(\Omega_+) + \frac{1}{9} H^4)
\]

in agreement with the Heterotic/IIA result
What about the six-dimensional (1,0) theory?

- The six dimensional Lagrangian supports our conjecture that only the odd-odd term picks up additional corrections beyond the introduction of a connection with torsion.
- Keeping only the NSNS sector corresponds to a (1,0) truncation.
- Suggests the supersymmetrization of (1,0) $R^2$ actions:
  \[
  t_4 t_4 R^2 \rightarrow t_4 t_4 R(\Omega_-)^2 + B \wedge \text{Tr} R(\Omega_-) \wedge R(\Omega_-) \\
  -\frac{1}{8} \epsilon_6 \epsilon_6 R^2 \rightarrow -\frac{1}{8} \epsilon_6 \epsilon_6 [R(\Omega_+)^2 + \frac{4}{3} H^2 R(\Omega_+) + \frac{1}{9} H^4] + B \wedge \text{Tr} R(\Omega_+) \wedge R(\Omega_+) 
  \]

- Supersymmetric Gauss-Bonnet in six dimensions?
Summary

▶ We have shown evidence that $R \rightarrow \mathcal{R}(\Omega_+)$ goes a long way towards completing the higher curvature corrections in string theory

▶ What is missing
  
  – The explicit six-dimensional can be lifted to ten dimensions, but this only goes up to $H^4 R^2$
  
  – We have focused on the NSNS sector only
  
  – Partial check on T-duality in the CP-even sector, but the expressions are long and the results are incomplete
  
  – Supersymmetry has played only a marginal role in what we have done
  
  – What about the tree-level higher derivative terms?
Further thoughts

- Can we determine the full eight-derivative invariant up to $H^8$?
  - A direct determination would require knowledge of the eight-point scattering amplitude

- What about the RR fields?
  - We may be able to lift to eleven dimensions, and the reduce to automatically determine the RR sector interactions

- Possible applications?
  - Avoiding no-go theorems for flux compactifications by turning on $H$ flux
  - Compactifications and lower-dimensional higher derivative couplings
  - Implications for AdS/CFT at finite $N$ and finite $\lambda$

- How does this tie in with generalized geometry and possible hidden symmetries of string theory?