Quantization of the Black Hole Horizons
A Discussion

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Texas AM University, April 8, 2013
A Quantization Condition?

• Many black holes satisfy the quantization condition:

\[ \frac{1}{(8\pi G_4)^2} A_+ A_- = \text{integer} . \]

Here \( A_+ \) is the usual area of the event horizon and \( A_- \) is the area of the *inner* horizon.

• *This is very surprising!*

• For example, the inner horizon is known to be an unstable Cauchy horizon; so how can its area be significant?

• The purpose of this talk: review and elaborate on aspects of the story.
Example: 4D Kerr

- Primary evidence for the quantization condition: inspection of explicit examples.
- This talk: focus on asymptotically flat spacetime in $D = 4$.
- The simplest example: the Kerr black hole.
- The “entropies” of Kerr black holes computed from outer and inner areas:

$$S_\pm = \frac{A_\pm}{4G_4} = 2\pi \left( G_4 M^2 \pm \sqrt{G_4^2 M^4 - J^2} \right).$$

- The product

$$\frac{1}{4\pi^2} S_+ S_- = J^2 = \text{integer}$$
Adding Charges

- In the context of $N = 4$ or $N = 8$ SUGRA the black holes can be generalized to carry arbitrary charges.

- For the outer and inner areas then give the product:

$$\frac{1}{(2\pi)^2} S_+ S_- = J^2 + J_4,$$

where the quartic invariant (for the $N = 4$ theory, for definiteness):

$$J_4 = \vec{Q}^2 \vec{P}^2 - (\vec{Q} \cdot \vec{P})^2.$$

- The right hand side of the product rule is an integer because $J_4$ is integral for correctly quantized charges.

- In fact: the right hand side is a \textbf{positive} integer since $J^2 > -J_4$ is the condition to avoid CTCs.
The Non-BPS Branch

- There is another branch of solutions with areas such that
  \[ \frac{1}{(2\pi)^2} S_+ S_- = -J_4 - J^2. \]

- Again, the right hand side is not only an integer, it is a positive integer since \( J_4 < -J^2 \) is the condition to avoid CTCs.

- The two black hole branches can be continuously deformed to each other, passing through the locus \( J^2 = -J_4 \).

- These transition solutions correspond to singular limits where \( A_- = 0 \) and the inner horizon has become singular.

- Example: the Schwarzschild solution.
Extremal Limits

• In the extremal limit the two horizons approach each other:
  \[ S_- \to S_+ . \]

• In this case
  \[ S_+ = 2\pi \sqrt{|J_4 + J^2|} . \]

• So: the integer in the quantization is the one that appears "under the square root" in the regime where extreme entropy is accounted for by the Cardy formula.

• The surprise is that this integer remains part of the story arbitrarily far from extremality.
Always an Integer?

- In the special case of a Kerr-Newman black hole:
  \[ \frac{1}{(2\pi)^2} S_+ S_- = \frac{1}{4} Q^4 + J^2. \]

- This is not generally an integer:
  \[ Q^2 = \alpha n_e^2 \quad \text{where} \quad \alpha^{-1} \simeq 137. \]

- Construction of Kerr-Newman as a solution in \( N \geq 2 \) SUGRA involves an embedding such as
  \[ Q_{KN} = \frac{1}{2} Q_{N=2} = \frac{1}{2} P_{N=2}. \]

- The Dirac quantization condition on the \( N = 2 \) theory effectively takes
  \[ Q_{N=2} P_{N=2} = 2\pi \times \text{integer} \Rightarrow \frac{1}{2} \alpha = \frac{e^2}{8\pi} = \text{integer}. \]
Generalized Attractor Mechanism

- The Right Hand Side of the quantization condition

\[ \frac{1}{(2\pi)^2} S_+ S_- = \text{integer} , \]

is independent of the black hole mass.

- It is also independent of scalar VEVs, i.e. on the position in moduli space.

- In particular, the relation can be continued from weak to strong coupling.

- This aspect can be thought of as a generalized attractor mechanism.
A First Law for the Inner Horizon

• "Thermodynamics" of the inner horizon must be considered in Lorentzian signature.

• For this Wald’s formalism works well (see later).

• The conservation of the Noether charge for the Killing generator of the inner horizon gives a "first law"

\[ T_- dS_ - dM - \Omega_- dJ - \Phi_- dQ - \Psi_- dP. \]

• In order to circumvent stability issues the differentials must be taken "in the space of solutions"; so there are no physical perturbations involved.

• The temperature \( T_- < 0 \) in the convention here (normal outward).
Consistency Relation

- Differentiation of the quantization rule with respect to mass gives:
  \[ T_+ S_+ = -T_- S_- \,.
  
  This is a strange relation that is highly nonobvious in concrete examples.

- Recall \( T_- < 0 \).

- Derivation of this consistency relation is tantamount to derivation of the quantization rule.
Interpretation of the Consistency Relation

• Suppose the general entropy can be divided into "chiral halves", as in 2D CFT:

\[ S_+ = S_L + S_R \equiv \frac{1}{2}(S_+ + S_-) + \frac{1}{2}(S_+ - S_-). \]

• The \( T_+ S_+ = -T_- S_- \) relation gives

\[ \frac{S_R}{T_R} = \frac{S_L}{T_L}. \]

• Interpretation: the central charge is the same in R and L sectors — so there is no diffeomorphism anomaly.
Another Consistency Relation?

• Concrete examples are also consistent with the general relation:

\[ \frac{\Omega_-}{T_-} = -\frac{\Omega_+}{T_+}. \]

This relation is equally bizarre.

• In the context of a CFT with two halves this relation amounts to

\[ \left( \frac{\partial S_L}{\partial J} \right)_M = 0. \]

• Interpretation: only R-movers in the 2D CFT have the ability to carry angular momentum.
Interpretation: Quantization Cond.

Model for the CFT (motivated by perturbative strings):

- Each chiral half has a quantized oscillator level $N_{L,R}$.
- Each has a zero mode contribution that depends on mass and moduli.
- Translational invariance imposes level matching:
  \[ N_L - N_R = J^2 + J_4. \]

Extremal limits:

- BPS states: $N_R = 0$ and $J = 0$ preserves SUSY.
- Kerr/CFT states: $N_R = 0$ and $J \neq 0$ breaks SUSY spontaneously.
- Non-BPS states: $N_L = 0$. 
Multistring Theory?

- DVV proposed a chiral multistring theory with level matching condition:
  \[ \sum_{k,l} k^{(l)} N^{(l)}_k = J_4. \]
  Here \( k \) is the momentum quantum number and \( l \) is the number of strings.

- A version of the theory is realized by the standard precision counting formulae for BPS black holes (the Igusa cusp form).

- The structure needed for non-extremal black holes would have both right and left movers.

- The general level matching condition can be approximated for large charges as
  \[ \sum k^{(l)} N^{(l)}_{L,k} - \sum k^{(l)} N^{(l)}_{R,k} = J^2 + J_4. \]
A GR Challenge

- The central observation is simply:

\[ A_+ A_- = \text{independent of mass} \iff T_+ S_+ + T_- S_- = 0 . \]

- It should be possible to prove this in classical GR!

- The following is an attempt in this direction (which unfortunately is not complete).
Noether-Wald Charge

- The variation of the Lagrangian $n$-form:
  \[ \delta L = E \delta \phi + d \Theta . \]

- $\phi$ denotes all fields, both metric ($g$) and matter fields ($\psi$).

- $E = 0$ is the equations of motion.

- The current $(n - 1)$-form corresponding to a diffeomorphism $\xi$ is
  \[ J = \Theta(\phi, \mathcal{L}_\xi \phi) - \xi \cdot L . \]

  Its divergence vanishes upon imposing the equations of motion:
  \[ dJ = -E \mathcal{L}_\xi \phi , \]

  Locally the current is thus the derivative of a Noether charge
  $(n - 2)$-form $Q$
  \[ J = dQ , \]
Killing Horizons

• The outer and inner horizons are Killing horizons of the Killing vectors

\[ \chi_\pm = \partial_t + \Omega_\pm \partial_\phi . \]

• The covariant derivatives of these Killing simplify at their respective horizons:

\[ \nabla^a \chi^b \bigg|_{\pm \text{hor}} = \pm \kappa_\pm \epsilon^{ab} , \]

where the bimetric satisfies \( g^{ab} = \epsilon^{ac} g_{cd} \epsilon^{cb} . \)

• The sign of the bimetric \( \epsilon^{ab} \) is such that it defines an outgoing normal at the outer horizon.
Killing Horizons

- For the Killing vectors, the total charges (after integration over the respective horizon at the bifurcation point):

\[ Q[\chi_{\pm}] = \int_{\text{hor}} X^{ab} \nabla_a \chi_{\pm b} = \pm \kappa_{\pm} A_{\pm} = \pm 8\pi G T_{\pm} S_{\pm} \]

- Charge conservation then gives the desired identity:

\[ T_+ S_+ + T_- S_- = 0 \]

- Well, almost.

- The “cross-terms” (the charges for \( \chi_{\pm} \) evaluated at the \( \mp \) horizon cancel in explicit computations but it is not clear that there is a general argument.

- Thus the argument is valid only for non-rotating black holes.
Summary

A discussion of an apparent quantization condition:

\[ \frac{1}{(8\pi G_4)^2} A_+ A_- = \text{integer} \]

Comments:

- The Right Hand Side does not depend on **black hole mass**.

- So it appears that there is some kind of index that can be **continued from extremality to non-extremality**.

- The Right Hand Side also does not depend on the **moduli** (VEVs of scalars at infinity)

- So it appears that the index can be **continued from weak to strong coupling** (before or after going to the extreme limit).