Rigid Supersymmetry in Curved Superspace

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Topics in Holography, Supersymmetry and Higher Derivatives

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Why Susy in curved space

Allows to compute exactly many interesting observables such as:

- Partition function Z on a compact manifold $\mathcal{M}$.
- Expectation value of supersymmetric operators.

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- Expectation value of supersymmetric operators.

For instance:

- The partition function on $S^3 \times S^1$ of $\mathcal{N} = 1$ theories with a $U(1)_R$ symmetry. [Romelsberger;. . .]
- The partition function on S^4 of $\mathcal{N} = 2$ theories. [Pestun;. . .]
- The partition function on S^3 of $\mathcal{N} = 2$ theories with a $U(1)_R$ symmetry.[Kapustin, Willett, Yaakov;. . .]

Why Susy in curved space

These results can be extended to less symmetric manifolds:

- $\mathcal{N} = 1, 2, 4$ on $S^3_b \times R$

- $\mathcal{N} = 2$ on S^3_b [Hama, Hosomichi, Lee; Imamura]
- $\mathcal{N} = 2$ on S^4_b [Hama, Hosomichi]

Why Susy in curved space

These results can be extended to less symmetric manifolds:

- $\mathcal{N} = 1, 2, 4$ on $S_b^3 \times R$
- $\mathcal{N} = 2$ on S_b^3 [Hama, Hosomichi, Lee; Imamura]
- $\mathcal{N} = 2$ on S_b^4 [Hama, Hosomichi]

Or to different number of dimensions:

- $\mathcal{N} = 2$ on $S^2 \times R$ [Kim;Imamura,Yokoyama]
- $\mathcal{N} = (2, 2)$ on S_b^2 [Benini, Cremonesi; Doroud, Gomis, Le Floch, Lee]
- $\mathcal{N} = 1$ on S^5 [Kallen,Qiu,Zabzine; Hosomichi, Seong, Terashima]
- $\mathcal{N} = (1, 0)$ on $S^3 \times S^3$ [Samtleben, Sezgin, Tsimpis]

There are interesting relations among these cases [Gadde, Yan; Benini, Nishioka, Yamazaki; ...]

Questions

- Which Riemannian manifolds $\mathcal{M}$ allow for Susy?
- What is the structure of supersymmetric theories on $\mathcal{M}$?
- Which observables can we compute exactly? What are their properties?

Outline

A general framework to understand Susy on curved manifolds.

Classification of Susy geometries

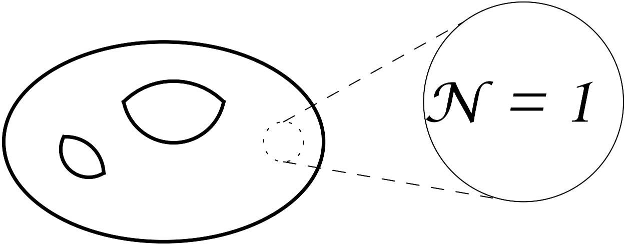
Conclusions

Susy on Curved Manifolds

Consider a supersymmetric theory in Euclidean flat space.

We want to place it on a Riemannian manifold $(\mathcal{M}, g)$ so that:

- The short distance limit of the theory is unaffected.
- The theory is invariant under some Supersymmetry.
- At short distances the Susy transformations are part of the flat space superalgebra.



A possible approach

- Start with a supersymmetric Lagrangian in flat space.
- Deform the metric to a curved manifold with size parametrized by r. Generically SUSY is broken.
- It is sometimes possible to preserve SUSY by deforming the SUSY variations of the fields and the Lagrangian order by order in $\frac{1}{r}$.

This procedure has several drawbacks:

- Case by case approach.
- Requires guesswork.
- The structure of the resulting theory is difficult to understand.

The Rigid Limit of SUGRA

Consider an off-shell formulation of Supergravity and give arbitrary background values to the fields in the gravity multiplet:

- The metric $g_{\mu\nu}$
- Various auxiliary fields.
- Set the gravitino $\psi_{\mu\alpha} = 0$

Send $M_p \to \infty$ keeping the background values for the metric and auxiliary fields fixed.

Some supersymmetry is preserved if it is possible to find ζ_α such that the SUSY variation of the gravitino is zero:

$$\delta_\zeta \psi_{\mu\alpha} = 0 \quad \Rightarrow \quad \nabla_\mu \zeta_\alpha = \mathcal{M}_{\mu\alpha}{}^{\beta} \zeta_\beta$$

where $\mathcal{M}_\mu$ depends on the the metric and auxiliary fields.

The Rigid Limit of SUGRA: Comments

- Different backgrounds treated in a unified way.
- Many terms in the Sugra Lagrangian drop out.
- Different than Linearized Sugra.
- We do not impose e.o.m. for the auxiliary fields. Different off shell formulations of SUGRA lead to distinct deformations.

The Rigid Limit of SUGRA: Comments

$$\nabla_\mu \zeta_\alpha = \mathcal{M}_{\mu\alpha}{}^{\beta} \zeta_\beta$$

- The "Killing" equation for ζ depends only on the fields in the gravity multiplet through ∇_μ and $\mathcal{M}_\mu$.
- We do not need to find backgrounds for matter and Sugra satifsying the e.o.m.

$$\Downarrow$$

There is no dependence on the matter content.

Generalized treatment of different theories.

New Minimal Sugra

In an $\mathcal{N} = 1$ theory with a $U(1)_R$ symmetry consider

- The energy momentum tensor $T_{\mu\nu}$
- The conserved R-current j^R_μ
- The supercurrent $S_{\mu\alpha}$

Together with a string current $C_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}\partial^\rho \mathcal{A}^\sigma$ they form the $\mathcal{R}$-multiplet. It couples to the fields in New Minimal Sugra:

- The metric $g_{\mu\nu}$
- The gravitino ψ^μ_α
- An auxiliary $U(1)_R$ connection A_μ
- A conserved vector V^μ

In the Rigid Limit we set $\psi^\mu_\alpha = 0$ and freeze the metric and auxiliary fields to arbitrary background values.

New Minimal Sugra, the Rigid Limit

Consider a flat space $\mathcal{N} = 1$ theory with an $U(1)_R$ symmetry. Coupling to Sugra and taking the rigid limit we obtain:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2$$

- $\mathcal{L}_0$ is the flat space theory minimally coupled to the metric.
- $\mathcal{L}_1$ are terms of order $\frac{1}{r}$ which couple the auxiliary fields to definite components of the R-multiplet

$$\mathcal{L}_1 = -j_\mu^{(R)} \left(A^\mu - \frac{3}{2} V^\mu \right) - \mathcal{A}_\mu V^\mu .$$

- At this order the deformation of the flat space theory can be described also when a Lagrangian is not available.
- $\mathcal{L}_2$ are $\frac{1}{r^2}$ terms with curvature or two auxiliary fields.

$$q \left(\frac{1}{4} R - \frac{3}{2} V_\mu V^\mu + 2 V_\mu A^\mu \right) (\phi\bar{\phi})$$

Rigid Variations

The Susy transformation of the matter are deformed from their flat space counterparts. E.g. for a chiral multiplet of R-charge q:

$$\begin{aligned}
\delta\phi^i &= -\sqrt{2}\zeta\psi^i \\
\delta\psi^i_\alpha &= -\sqrt{2}\zeta_\alpha F^i - i\sqrt{2}(\sigma^\mu\bar{\zeta})_\alpha(\partial_\mu - iqA_\mu)\phi^i \\
\delta F^i &= -i\sqrt{2}\bar{\zeta}\bar{\sigma}^\mu\left(\nabla_\mu - i(q-1)A_\mu - \frac{i}{2}V_\mu\right)\psi^i
\end{aligned}$$

Setting to zero the gravitino variation gives the Killing spinor equations:

$$\begin{aligned}
(\nabla_\mu - iA_\mu)\zeta &= -iV_\mu\zeta - iV^\nu\sigma_{\mu\nu}\zeta \\
(\nabla_\mu + iA_\mu)\bar{\zeta} &= iV_\mu\bar{\zeta} + iV^\nu\bar{\sigma}_{\mu\nu}\bar{\zeta}
\end{aligned}$$

On an Euclidean manifold $\mathcal{M}$ the spinors ζ and $\bar{\zeta}$ are independent and V_μ, A_μ are complex.

3D New Minimal Sugra

$\mathcal{N}=1$ theories in 4D reduce to 3D $\mathcal{N}=2$ theories.

$$\{Q_\alpha, \bar{Q}_\beta\} = 2\gamma^\mu_{\alpha\beta} P_\mu + 2i\epsilon_{\alpha\beta} Z \ , \quad \{Q_\alpha, Q_\beta\} = 0 \ , \quad \{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0$$

The $\mathcal{R}$-multiplet reduces to: [Dumitrescu,Seiberg]

$$T^{\mu\nu}, \quad S^\mu_\alpha, \quad j^R_\mu, \quad j^Z_\mu, \quad J$$

New Minimal Sugra in $3D$ has three auxiliary fields with couplings:

$$j^R_\mu \left(A^\mu - \frac{3}{2} V^\mu \right) - i\, j^Z_\mu C^\mu + JH, \qquad V = *dC$$

For a Superconformal theory j^Z_μ and J are redundant operators.

3D New Minimal Sugra

A component formulation of $\mathcal{N} = 2$ New Minimal Sugra in 3d is not available.

We can however understand its rigid limit using:

- Linearized Sugra around flat space.
- Dimensional reduction of 4d New Minimal Sugra.

We find the Killing Spinor equations

$$\begin{aligned}(\nabla_\mu - iA_\mu)\zeta &= -\tfrac{1}{2}H\gamma_\mu\zeta - iV_\mu\zeta - \tfrac{i}{2}\epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\zeta \\ (\nabla_\mu + iA_\mu)\tilde{\zeta} &= -\tfrac{1}{2}H\gamma_\mu\tilde{\zeta} + iV_\mu\tilde{\zeta} + \tfrac{i}{2}\epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\tilde{\zeta}\end{aligned} \tag{1}$$

Matter multiplets and Susy Lagrangians can also be derived.

Example S^3 [Kapustin,Willet,Yaakov; Jafferis,...]

$\mathcal{N}=2$ theory with $\mathrm{U}(1)_R$ on S^3 is SUSY for

$$H=-\frac{i}{r}, \qquad V=A=0$$

There are 4 supercharges with superalgebra $SU(2|1)_\ell \times SU(2)_r$

Not reflection positive unless H decouples (theory is conformal).

When additional $U(1)_f$ flavor symmetries are present

- We can add real masses m_f
- $\mathrm{U}(1)_R$ can be shifted by $\mathrm{U}(1)_f$ (Improvements of $\mathcal{R}$ multiplet)

$$j_\mu^R \to j_\mu^R + t\, j_\mu^f\ , \ldots$$

- They appear as $t+i\ m_f$

Classification of

Supersymmetric manifolds

The rigid limit of Sugra allows to systematically classify on which manifolds it is possible to define Susy field theories.

Weak dependence on the matter content of the field theories: only relevant information is the nature of the supercurrent multiplet.

- $\mathcal{N} = 1$ theories with a $U(1)$ R-symmetry in 4d [Klare, Tomasiello, Zaffaroni; Dumitrescu, Seiberg, GF]
- $\mathcal{N} = 1$ theories with an FZ supercurrent multiplet in 4d [Samtleben, Tsimpis; Dumitrescu, GF]
- $\mathcal{N} = 2$ theories with a $U(1)$ R-symmetry in 3d [Klare, Tomasiello, Zaffaroni; Closset, Dumitrescu, GF, Komargodski]
- $\mathcal{N} = (1, 0)$ theories in 6d [Samtleben, Sezgin, Tsimpis]

Also see [Cassani, Klare, Martelli, Tommasiello, Zaffaroni; Hristov, Tomasiello, Zaffaroni] for analyses in Lorentzian signature.

Classifying Supersymmetric Manifolds

Consider the 4d New Minimal Sugra Killing spinor equation.

On which Riemannian Manifolds $(\mathcal{M}, g)$ are there solutions of

$$(\nabla_\mu - iA_\mu)\zeta = -iV_\mu\zeta - iV^\nu\sigma_{\mu\nu}\zeta$$
$$(\nabla_\mu + iA_\mu)\bar{\zeta} = iV_\mu\bar{\zeta} + iV^\nu\bar{\sigma}_{\mu\nu}\bar{\zeta}$$

for some choice of background fields A_μ and V_μ?

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- Two Killing spinors of opposite R-charge ζ and $\bar{\zeta}$ are present only on torus fibrations over a Riemann surface Σ.
- Two Killing spinors of the same R-charge require $SU(2)$ holonomy (compact case).
- Four supercharges are only present on $S^3 \times R$ or $H^3 \times R$ (and their compactifications)

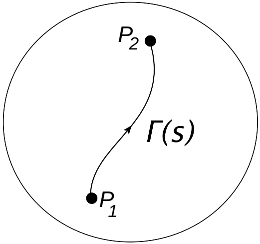
One supercharge in 4d with $U(1)_R$

$$(*) \qquad (\nabla_\mu - iA_\mu)\zeta = -iV_\mu\zeta - iV^\nu\sigma_{\mu\nu}\zeta$$

- ζ has R-charge 1
- $\zeta \in (1/2, 0)$ in $SU(2)_l \times SU(2)_r$

Connect P_1 and P_2 by a curve $\Gamma(s)$
On Γ (*) reduces to an ODE which is linear first order and homogeneous:

$$\frac{d}{ds}\zeta_\alpha = \mathcal{P}(s)_\alpha{}^\beta \zeta_\beta$$



The metric $g_{\mu\nu}$ and A_μ, V_μ are smooth and nonsingular.
Hence a nontrivial solution ζ is nowhere vanishing.

One supercharge in 4d with $U(1)_R$

Starting with a solution ζ we can form the tensor

$$J^{\mu}{}_{\nu} = \frac{2i}{|\zeta|^2}\zeta^{\dagger}\sigma^{\mu}{}_{\nu}\zeta$$

- $J^{\mu}{}_{\nu}$ is an almost complex structure $J^{\mu}{}_{\nu}J^{\nu}{}_{\rho} = -\delta^{\mu}_{\rho}$
- $J^{\mu}{}_{\nu}$ is metric compatible.
- $J^{\mu}{}_{\nu}$ is integrable because ζ is Killing. (satisfies (*))

The triple $(\mathcal{M},\ g_{\mu\nu},\ J^{\mu}{}_{\nu})$ defines an Hermitian manifold.

One supercharge in 4d with $U(1)_R$

The complex structure is not covariantly constant under the Levi-Civita connection (unless $\mathcal{M}$ is Kähler)

$$V_\mu = -\frac{1}{2}\nabla_\nu J^\nu{}_\mu + U_\mu, \quad J^\mu{}_\nu U^\nu = i U^\nu, \quad \nabla^\mu U_\mu = 0$$

On any Hermitian manifold we have connections (with torsion) such that

- $\nabla^c_\mu g_{\nu\rho} = \nabla^c_\mu J^\nu{}_\rho = 0$
- The holonomy of ∇^c_μ is $U(1)_l \times SU(2)_r \subset SU(2)_r \times SU(2)_l$

Using ∇^c_μ we rewrite the Killing spinor equation as

$$(\nabla^c_\mu - iA^c_\mu)\zeta = 0, \qquad A^c_\mu = A_\mu - \frac{1}{2}(\delta^\nu_\mu - iJ_\mu{}^\nu)V_\nu - \frac{1}{2}U_\mu$$

We can use A^c_μ to twist away the $U(1)_l$ holonomy and find a Killing spinor ζ on any complex manifold.

One supercharge in 4d with $U(1)_R$

On a Kähler manifold $V_\mu = 0$ and we only need to turn on A_μ.

The Superalgebra generated by ζ is $\{Q_\zeta, Q_\zeta\} = 0$.

Open questions:

For a compact complex manifold what does $Z(\mathcal{M})$ depend on?

- Choice of metric
- Choice of complex structure

One supercharge in 3d

There is a similar story in 3d

$$(*) \quad (\nabla_\mu - iA_\mu)\zeta = -\frac{1}{2} H\gamma_\mu\zeta - iV_\mu\zeta - \frac{i}{2}\epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\zeta$$

Again a nontrivial solution has no zeros and we define:

$$\eta^\mu = \frac{1}{|\zeta|^2}\zeta^\dagger\gamma^\mu\zeta, \qquad \Phi^\mu{}_\nu = \epsilon^\mu{}_{\nu\rho}\eta^\rho$$

- $\eta_\mu\eta^\mu = 1$
- $g_{\mu\nu} = \eta_\mu\eta_\nu - \Phi_{\mu\rho}\Phi^\rho{}_\nu$

This defines an almost contact metric structure on $\mathcal{M}$
Because of the Killing equation (*) it satisfies the condition

$$\Phi^\mu{}_\rho\mathcal{L}_\eta\Phi^\rho{}_\nu = 0$$

One supercharge in 3d

This condition is similar to the integrability condition for $J^{\mu}{}_{\nu}$.

We can choose coordinates $\tau \in R$ and $z \in C$ such that

- $\eta^{\mu}\partial_{\mu} = \partial_{\tau}$
- $\Phi^{z}{}_{\bar{z}} = \Phi^{\bar{z}}{}_{z} = 0$ while $\Phi^{z}{}_{z} = i$ and $\Phi^{\bar{z}}{}_{\bar{z}} = -i$
- $ds^2 = (d\tau + h(\tau, z, \bar{z})dz + \bar{h}(\tau, z, \bar{z})d\bar{z})^2 + c(\tau, z, \bar{z})\, dz d\bar{z}$
- Two such coordinate charts are related by

$$\tau' = \tau + t(z, \bar{z}), \qquad z' = f(z)$$

Given a metric $g_{\mu\nu}$ on an orientable three manifold $\mathcal{M}$ it is always possible to find η^{μ} which satisfies the constraint in a patch.
However there can be global obstructions.

One supercharge in 3d

As in 4D there are connections (with torsion) such that

- $\nabla^c_\mu g_{\nu\rho} = \nabla^c_\mu \eta_\nu = 0$
- The holonomy of ∇^c_μ is contained in $U(1) \subset SU(2)$

Given a $(\Phi^\mu{}_\nu,\ \eta_\mu)$ satisfying the constraint we can rewrite the Killing spinor equation as

$$(\nabla^c_\mu - iA^c_\mu)\zeta = 0, \qquad A^c_\mu = A_\mu + \ldots$$

Again we use A^c_μ to twist away the $U(1)$ holonomy and find a Killing spinor ζ.

H and V_μ are determined by $\nabla_\mu \eta_\nu$.

Two Supercharges in 4d

If a second solution $\bar{\zeta}$ is present there are further restrictions on the metric. Consider the complex vector field

$$K^\mu = \bar{\zeta}\bar{\sigma}^\mu\zeta, \qquad Re(K^\mu) = X^\mu, \qquad Im(K^\mu) = Y^\mu$$

- K^μ is Killing.
- $J^\mu{}_\nu$ is determined by K^μ and the metric.
- If $[X, Y] \neq 0$ the manifold is locally isometric to $S^3 \times R$
- If $[X, Y] = 0$ the two Killing vector fields X and Y generate translations on a T^2 fibered over a Riemann surface Σ

Two Supercharges 4d/3d

The superalgebra generated by ζ and $\bar{\zeta}$ is

$$\{Q_\zeta, Q_\zeta\} = 0, \quad \{Q_{\bar{\zeta}}, Q_{\bar{\zeta}}\} = 0, \quad \{Q_{\bar{\zeta}}, Q_\zeta\} = \delta_K$$

$$[\delta_K, Q_\zeta] = [\delta_K, Q_{\bar{\zeta}}] = 0$$

By reducing along one direction on the T^2 we obtain the following:

Any $\mathcal{N} = 2$ field theory with a $U(1)_R$ symmetry in 3d can be placed on a circle bundle over Σ preserving two supercharges.

All squashed 3-spheres in the literature are in this class.

Question: can we apply localization techniques to study supersymmetric field theories on these manifolds?

Conclusions

Turning on background values for the fields in the supergravity multiplet and taking the rigid limit allows a general description of rigid SUSY in curved superspace.

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See [Samtleben, Sezgin, Wimmer] for a 6d analysis.

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Several new question to be addressed...

Thank You!