Chiral Liouville Gravity & New View on $AdS_3$

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References

Preambule

Two dimensional critical behavior is often associated with the emergence of infinite-dimensional conformal symmetries

\[ \delta t^+ = \epsilon^+(t^+), \quad \delta t^- = \epsilon^-(t^-), \]

forming two copies of the (centerless) Virasoro algebra.

In the last years, evidence has been accumulating for another possible critical behavior associated with the so-called “warped conformal symmetries”

\[ \delta t^+ = \epsilon(t^+), \quad \delta t^- = \sigma(t^+), \]

which act in a chiral fashion. It forms a semi-direct product of a Virasoro algebra with a \( U(1) \) Kac-Moody algebra.
The two-dimensional Virasoro $\times$ Virasoro algebra is represented both as a

- Symmetry of conformal field theories (CFTs)
- Subgroup of diffeomorphisms of gravitational theories which acts non-trivially

In some cases, the two representations are related by the AdS/CFT correspondence.
In the last years, the Virasoro $\times$ Kac-Moody algebra has appeared in various related contexts as

- the asymptotic symmetry algebra of warped $AdS_3$ spacetimes
  [G.C., Detournay, 2008]

- Generalizations of the Kerr/CFT correspondence [Guica, Hartman, Song, Strominger, 2008]

- an alternative symmetry enhancement of unitary QFTs with $SL(2, \mathbb{R}) \times U(1)$ symmetry [Hofman, Strominger, 2011]

- a sufficient symmetry to derive the entropy of warped $AdS_3$ black holes [Detournay, Hartman, Hofman, 2012]

However, no clear example of field theory or gravitational theory was proposed where warped symmetry is realized.

Given the universality of CFTs and their importance in holography, theories with warped conformal symmetries, if they exist, are expected to also play an important role.
In this talk, I will present a simple and tractable example where the warped symmetry is realized: chiral Liouville gravity.

The theory turns out to be consistent at the classical and semi-classical level.

The existence of a quantum theory is left for further work.

I will also show that this theory is dual to pure Einstein gravity in $AdS_3$ with appropriate boundary conditions.

In that sense, gravity in $AdS_3$ is not always dual to a $CFT$. It can also be dual to a warped $CFT$, depending on the boundary conditions.
Outline

1. Preambule
2. Chiral Liouville gravity
3. Boundary theory of $AdS_3$ Einstein gravity
Polyakov gravity and Liouville theory

The Polyakov action for $1 + 1$ quantum gravity is

$$S_P = \frac{c}{96\pi} \int d^2x \sqrt{-g} (R \Box^{-1} R - 2\Lambda)$$

or, more precisely,

$$S_P = \frac{c}{96\pi} \left( \int d^2x \int d^2x' \mathcal{R}(x) G(x, x') \mathcal{R}(x') - 2\Lambda \int d^2x \sqrt{-g} \right),$$

where

$$\mathcal{R} \equiv \sqrt{-g} R,$$

$$\sqrt{-g} g^{ab} \nabla_a \nabla_b G(x, x') = \delta^2(x, x').$$
The stress tensor defined by the metric variations of this action is

\[ T_{ab} \equiv \frac{2\pi}{\sqrt{-g}} \frac{\delta S_P}{\delta g^{ab}}. \]

The trace of the stress tensor is the local expression

\[ T^a_a = \frac{c}{24} (R + \Lambda). \]

In order to make sense of the theory, one usually gauge fixes part of the diffeomorphisms.

This is very subtle because diffeomorphisms are all pure gauge. They might act non-trivially.
Upon fixing the metric to light-cone “gauge”

\[ g_{--} = 0 = g_{++}, \]

\[ -\frac{1}{2} e^{2\rho} \equiv g_{+-}, \]

the action reduces to the local Liouville action

\[ S_P = \frac{c}{12\pi} \int dt^+ dt^- \left( \partial_- \rho \partial_+ \rho - \frac{\Lambda}{8} e^{2\rho} \right). \]

Conformal symmetries preserves the light-cone gauge. They act non-trivially on \( \rho \). Therefore, Liouville theory is a CFT.

The theory has been quantized. The theory is integrable.
Chiral Liouville gravity

We consider the chiral gauge fixing,

\[ g_{--} = 0, \quad \partial_-(g^{+-}g_{++}) = 0. \]

The metric is

\[ ds^2 = -e^{2\rho}(dt^+dt^- - h(dt^+)^2). \]

\[ \partial_- h = 0. \]

For \( h \neq 0 \), the gauge fixing is invariant under the warped conformal symmetries

\[ \delta t^+ = \epsilon(t^+), \quad \delta t^- = \sigma(t^+). \]

The diffeomorphisms \( \delta t^- = \epsilon^-(t^-) \neq \text{cst} \) are not allowed.
We consider the Polyakov action with an additional chiral term,

\[
\frac{c}{96\pi} \int d^2x \sqrt{-g}(R \Box^{-1} R - 2\Lambda) + \frac{\Delta}{4\pi} \int d^2x \sqrt{-g}g^{--}.
\]

The chiral boundary term is invariant under warped conformal symmetries but not under standard conformal symmetries.

Using the equation of motion for \(g^{--}\), it ensures that the left-moving stress-tensor zero mode is a fixed constant

\[
T_{--} + \frac{\Delta}{2} = 0.
\]

This defines chiral Liouville theory at the classical level.
Chiral Liouville action

Using the gauge fixing condition $g_{--} = 0$, the action reduces to

$$S = \frac{c}{12\pi} \int dt^+ dt^- \left( \partial_+ \rho \partial_- \rho - \frac{\Lambda}{8} e^{2\rho} + h(\partial_- \rho)^2 - \frac{6}{c} h \Delta + \partial_- h \partial_- \rho + \frac{1}{2} \int d^2 x' \partial_- h(x) G(x, x') \partial_- h(x') \right).$$

Using the gauge fixing condition $\partial_-(g^{+-} g_{++}) = \partial_- h = 0$, the action becomes local

$$S = \frac{c}{12\pi} \int dt^+ dt^- \left( \partial_+ \rho \partial_- \rho - \frac{\Lambda}{8} e^{2\rho} + h \left( (\partial_- \rho)^2 - \frac{6}{c} \Delta - \partial_-^2 \rho \right) \right).$$

One can define the quantum theory at the semi-classical level using canonical quantization.
Conserved currents

The Noether procedure leads to the currents

\[ j_\epsilon^a = e^{2\rho} T^a_\epsilon, \quad j_\sigma^a = e^{2\rho} T^a_\sigma. \]

where the stress-tensor is

\[ T_{ab} = \frac{2\pi}{\sqrt{-g}} \frac{\delta L_P}{\delta g^{ab}} + \Delta \begin{pmatrix} h^2 & -\frac{3}{2}h \\ -\frac{h}{2} & \frac{1}{2} \end{pmatrix}_{ab}. \]

The full \( T_{ab} \) is not symmetric because of the asymmetric chiral action.

After imposing \( T_{\epsilon\epsilon} = 0, j_\epsilon^+ = j_\sigma^+ = 0 \) and we are left with two right-moving conserved currents

\[ \partial_- j_\epsilon^- = 0 = \partial_- j_\sigma^- . \]
Warped conformal symmetry algebra

Using canonical quantization, one can compute Dirac brackets.

The current-current commutators are obtained as

\[
[j^-_\sigma(t^+), j^-_\sigma(s^+)] = \pi k \partial_{t^+} \delta(t^+ - s^+),
\]
\[
[j^-_\epsilon(t^+), j^-_\sigma(s^+)] = 2\pi j^-_\sigma(t^+) \partial_{t^+} \delta(t^+ - s^+)
\]
\[
[j^-_\epsilon(t^+), j^-_\epsilon(s^+)] = 2\pi \partial_{t^+} \delta(t^+ - s^-) (j^-_\epsilon(t^+) + j^-_\epsilon(s^+)) - \frac{\pi c}{6} \partial_{t^+}^3 \delta(t^+ - s^+).
\]

This is a Virasoro Kac-Moody algebra with central charge \( c \) and level

\[ k = -4\Delta. \]
Chiral Liouville gravity is integrable

The Bäcklund transformation is

$$e^{2\rho} = -\frac{8}{\Lambda} \sqrt{\frac{6\Delta}{c}} e^{2O} \left( \cosh \left( \sqrt{\frac{6\Delta}{c}} (t^- - P) \right) \right. $$

$$- \sinh \left( \sqrt{\frac{6\Delta}{c}} (t^- - P) \right) \int dt^+ e^{2O} \right)^{-2},$$

$$h = \partial_+ P,$$

The equations of motion $R = -\Lambda$ and $T_- = 0$ then reduce to the free field equations

$$\partial_- O = 0, \quad \partial_- P = 0.$$

Evaluating the currents reveals

$$j_\epsilon^- = \frac{c}{6} \left( (\partial_+ O)^2 - \partial_+^2 O \right) - \Delta (\partial_+ P)^2,$$

$$j_\sigma^- = 2\Delta \delta^{-+} \partial_+ P.$$
Warped CFT as a deformed CFT

The chiral Liouville action is

\[
S = \frac{c}{12\pi} \int dt^+ dt^- \left( \partial_+ \rho \partial_- \rho - \frac{\Lambda}{8} e^{2\rho} + h \left[ (\partial_- \rho)^2 - \frac{6}{c} \Delta - \partial_-^2 \rho \right] \right).
\]

The first two terms define the Liouville CFT.

Since \( h = j^\sigma_\sigma \), it has conformal weight \((1, 0)\). \( \partial_-^2 \rho \) and \((\partial_- \rho)^2\) have conformal weight \((0, 2)\).

Therefore, the CFT is deformed by a \((1, 2)\) operator. This is connected to so-called dipole deformations.

This connection has not yet been understood.
Coupling to Matter

Take any QFT, e.g. a scalar field

\[
\int d^2x \left( \partial_\mu \Phi \partial^\mu \Phi + m^2 \Phi^2 \right).
\]

One can define a new warped CFT by

- Making the action generally covariant and adding the Polyakov action
- Fixing to chiral gauge and fixing \( T^- = \frac{\Delta}{2} \)

For the scalar field one obtains

\[
S = S_P + \int dt^+ dt^- (\partial_+ \Phi \partial_- \Phi + h(\partial_- \Phi)^2 - \frac{m^2}{4} e^{2\rho} \Phi^2).
\]

The resulting theory is a warped CFT.
Summary

- The Polyakov action in chiral gauge leads to Chiral Liouville Gravity

- The theory forms a non-trivial representation of the warped conformal symmetry

\[ \delta t^+ = \epsilon(t^+), \quad \delta t^- = \sigma(t^+) \]

\[ \text{and is therefore a warped CFT.} \]

- The theory can be defined semi-classically.

- The theory is integrable.

- An infinite number of warped CFTs can be built from coupling Chiral Liouville Gravity to any matter QFT.
Claim:

$1 + 1$ Chiral Liouville gravity is holographically equivalent to $AdS_3$ Einstein gravity with new chiral boundary conditions, at the semi-classical level.
Milestones of $AdS_3$ Einstein gravity

1986, Brown-Henneaux: Consistent boundary conditions with Virasoro $\times$ Virasoro asymptotic symmetries.

1993, Banados, Teitelboim, Zanelli: $AdS_3$ Black holes

1995, Coussaert, Henneaux, van Driel: $AdS_3$ Chern-Simons gravity with Brown-Henneaux boundary conditions is equivalent to Liouville theory

1998, Strominger: BTZ entropy counting from dual CFT

2007, Maloney, Witten: No consistent quantum $AdS_3$ Einstein gravity

The role of Brown-Henneaux boundary conditions is crucial.

Now, another set of boundary conditions exist.
Derivation of boundary conditions

The most general solution of Einstein gravity can be written as

\[ ds^2 = \frac{dr^2}{r^2} + r^2 (g_{ab}^{(0)} + \frac{1}{r^2} g_{ab}^{(2)} + \ldots) \]

where

\[ D_a^{(0)} g_{ab}^{(2)} \sim \partial_b R^{(0)}, \quad g_{ab}^{(0)} g_{ab}^{(2)} \sim R^{(0)}. \]

Assuming no conformal anomaly \( R^{(0)} = 0 \) the variation of the Einstein action including covariant counterterms is

\[ \delta S \sim \int d^2 x \ g_{ab}^{(2)} \delta g_{ab}^{(0)}. \]
Known boundary conditions

Consistent boundary conditions only require

$$\delta S \sim \int d^2x \, g^{ab}_{(2)} \delta g^{(0)}_{ab} = 0.$$ 

Boundary conditions known before 2013:

- **Dirichlet [Brown-Henneaux]** \( g^{(0)}_{ab} = \bar{\eta}_{ab} \).
- **Neumann [G.C., Marolf]** \( g^{ab}_{(2)} = 0 \).

New boundary conditions:

- **Chiral boundary conditions [G.C., Song, Strominger]**

  \[ g^{(0)}_{--} = 0, \quad g^{(0)}_{+-} = -\frac{1}{2}, \quad g^{++}_{(2)} = \frac{16G}{\ell} \Delta. \]

- **Generalized Dirichlet [Troessaert]**

  \[ g^{(0)}_{ab} = e^{2\phi} \bar{g}_{ab}, \quad \delta \bar{g}_{ab} = 0. \]
Chiral Boundary conditions

The boundary conditions can be written as

\[ g^{(0)}_{--} = 0, \quad g^{(0)}_{+-} = -\frac{1}{2}, \quad g^{(0)}_{++} = \partial_+ \bar{P}(t^+), \]

\[ g^{++}_{(2)} = \frac{16G}{\ell} \Delta. \]

The total action is defined as the Einstein action with Gibbons-Hawking and covariant counterterms \( S_0 \) complemented by a chiral boundary term,

\[ S = S_0 + \frac{\Delta}{4\pi} \int dt^+ dt^- \sqrt{-g^{(0)}_{(0)} g^{--}_{(0)}}. \]
Phase space

The most general solution with chiral boundary condition contains the BTZ black holes with $\ell M = \Delta + \bar{\Delta}$ and $J = \Delta - \bar{\Delta}$

$$\frac{ds^2}{\ell^2} = \frac{dr^2}{r^2} - r^2 dt^+ dt^- + \frac{\bar{\Delta}}{k} (dt^+)^2 + \frac{\Delta}{k} (dt^-)^2 - \frac{\Delta \bar{\Delta}}{k^2 r^2} dt^+ dt^-,$$

dressed by boundary gravitons ($\bar{L}(t^+)$) and boundary photons ($\bar{P}(t^+)$). Here $k = \ell/4G$. It can be written as

$$ds^2 = \frac{\ell^2}{r^2} dr^2 - \ell^2 r^2 dt^+ (dt^- - \partial_+ \bar{P}(t^+) dt^+)$$

$$+ 4G\ell \left[ \bar{L}(t^+) (dt^+)^2 + \Delta (dt^- - \partial_+ \bar{P}(t^+) dt^+)^2 \right]$$

$$- \frac{16G^2 \Delta}{r^2} \bar{L}(t^+) dt^+ (dt^- - \partial_+ \bar{P}(t^+) dt^+).$$
Its form is preserved by the Kac-Moody-Virasoro asymptotic transformations

\[ \delta t^+ = \epsilon(t^+), \quad \delta t^- = \sigma(t^+), \]

which generate boundary gravitons \( \bar{L}(t^+) \) and photons \( \bar{P}(t^+) \).

The associated charges are

\[
Q_\epsilon = \frac{1}{2\pi} \int_0^{2\pi} d\phi \, \epsilon(t^+) \left( \bar{L}(t^+) - \Delta (\partial_+ \bar{P}(t^+))^2 \right),
\]

\[
Q_\sigma = \frac{1}{2\pi} \int_0^{2\pi} d\phi \, \sigma(t^+) (\Delta + 2\Delta \partial_+ \bar{P}(t^+)).
\]

The Virasoro charge contains a Sugawara term.
Warped conformal asymptotic symmetry

Expanding in modes $e^{int^+}$ we obtain the Dirac brackets

$$i\{\mathcal{L}_m, \mathcal{L}_n\} = (m - n)\mathcal{L}_{m+n} + \frac{c_R}{12}m^3\delta_{m,-n},$$

$$i\{\mathcal{L}_m, \mathcal{J}_n\} = -n\mathcal{J}_{m+n},$$

$$i\{\mathcal{J}_m, \mathcal{J}_n\} = \frac{k_{KM}}{2}m\delta_{m,-n}.$$

The Virasoro central charge and level of the $U(1)$ Kac-Moody algebra are given by

$$c_R = \frac{3\ell}{2G}, \quad k_{KM} = -4\Delta.$$

- $AdS_3$ ground state well-defined.
- Superradiance effect around black holes.
Lagrangian reduction

In order to dimensionally reduce $AdS_3$ Einstein gravity as a boundary theory, we switch to the Chern-Simons formulation,

\[ S_E[A, \bar{A}] = S_k[A] + S_{-k}[\bar{A}] + \text{(Boundary terms)} \]

where

\[ S_k[A] = \frac{k}{4\pi} \int_{\mathcal{M}} \text{Tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A) \]

and $k = \frac{\ell}{4G}$.

Boundary terms are fixed by the boundary conditions.
The chiral boundary conditions read as

\[
\begin{align*}
A &= \begin{pmatrix}
\frac{d}{2r}r(dt^- - \partial_+ \bar{P}(t^+)dt^+) & \frac{\Delta}{kr} (dt^- - \partial_+ \bar{P}(t^+)dt^+) \\
0 & -\frac{d}{2r}
\end{pmatrix} \\
&\quad + \begin{pmatrix}
O(r^{-2}) & O(r^{-2}) \\
O(r^{-1}) & O(r^{-2})
\end{pmatrix},
\end{align*}
\]

\[
\bar{A} = \begin{pmatrix}
-\frac{d}{2r} + O(r^{-2}) & rdt^+ + O(r^{-2}) \\
\frac{1}{kr} \bar{L}(t^+)dt^+ + O(r^{-2}) & \frac{d}{2r} + O(r^{-2})
\end{pmatrix}.
\]

The chiral boundary conditions differ from the Brown-Henneaux ones only in the left sector.

We will only discuss the left sector. The right sector gives a chiral right-moving boson.
The equations $F = 0$ can be solved consistently with the boundary conditions by

$$A = G_L^{-1} dG_L, \quad G_L = g_L(t^+, t^-) \begin{pmatrix} \sqrt{r} & 0 \\ 0 & \frac{1}{\sqrt{r}} \end{pmatrix},$$

where the Gauss decomposition gives

$$g_L = \begin{pmatrix} 1 & 0 \\ Y_L & 1 \end{pmatrix} \begin{pmatrix} e^{-\frac{\Phi_L}{2}} & 0 \\ 0 & e^{\frac{\Phi_L}{2}} \end{pmatrix} \begin{pmatrix} 1 & X_L \\ 0 & 1 \end{pmatrix}.$$ 

The action is equivalent to

$$S_L = \frac{k}{8\pi} \int_\Sigma dt^+ dt^- [-(\partial_+ \Phi_L \partial_- \Phi_L + 4e^{-\Phi_L} \partial_+ X_L \partial_- Y_L)$$

$$+8 \frac{\Delta}{k} \frac{\partial_+ X_L}{\frac{\Delta}{k} - X_L^2}].$$
The boundary conditions imply further constraints among the fields $X_L, Y_L, \Phi_L$ :

\begin{align*}
\partial_+ X_L &= (X_L^2 - \frac{\Delta}{k}) \partial_+ \bar{P}, \\
\partial_- X_L &= \frac{\Delta}{k} - X_L^2, \\
\partial_- \Phi_L &= -2X_L, \\
\partial_+ \Phi_L &= 2\partial_+ \bar{P}X_L, \\
\partial_+ Y_L &= -e^{\Phi_L} \partial_+ \bar{P}, \\
\partial_- Y_L &= e^{\Phi_L}.
\end{align*}

The general solution is two chiral bosons determined by $\bar{P}$ and $Y_L$. The right chiral boson is determined by $X_R$. 
Combination of left and right sectors

We define the combined fields

\[ e^\Phi = \frac{\partial_- Y_L \partial_+ X_R}{(1 - Y_L X_R)^2}, \quad h = \partial_+ \bar{P} \]

The constraints are equivalent to the equations of motion of the chiral Liouville action

\[ S = \frac{k}{8\pi} \int dt^+ dt^- \left( \partial_+ \Phi \partial_- \Phi + 4e^\Phi + h((\partial_- \Phi)^2 - 2\partial_-^2 \Phi - \frac{4\Delta}{k}) \right), \]

\[ \partial_- h = 0. \]

Moreover, the Dirac brackets of the Virasoro-Kac-Moody generators are equivalent.

Therefore, the Lagrangian reduction of \( AdS_3 \) gravity is equivalent to chiral Liouville gravity.
Conclusion

- We obtained the first $1 + 1$ Warped Conformal Field Theory with Virasoro $\times U(1)$ Kac-Moody symmetry.
- Chiral Liouville theory can be obtained from Polyakov gravity in chiral gauge.
- It can also be obtained holographically from $AdS_3$ Einstein gravity with chiral boundary conditions.
- The theory is integrable and defined semi-classically.
- The theory can be coupled to matter, as ordinary Liouville theory can.

Important question left: what about the quantum theory?

Other question: what does it tell us about black holes?