Chiral Liouville Gravity & New View on AdS_3

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Holography and its Applications,
Texas A&M, College Station, April 8-12th

References

- “Chiral Liouville Gravity”,
  G.C., W. Song and A. Strominger, arXiv :1303.2662
- “New Boundary Conditions for AdS_3,
  G.C., W. Song and A. Strominger, arXiv :1303.2660

Preamble

Two dimensional critical behavior is often associated with the emergence of infinite-dimensional conformal symmetries

$$\delta t^+ = \epsilon^+(t^+), \qquad \delta t^- = \epsilon^-(t^-).$$

forming two copies of the (centerless) Virasoro algebra.

In the last years, evidence has been accumulating for another possible critical behavior associated with the so-called "warped conformal symmetries"

$$\delta t^+ = \epsilon(t^+), \qquad \delta t^- = \sigma(t^+),$$

which act in a chiral fashion. It forms a semi-direct product of a Virasoro algebra with a $U(1)$ Kac-Moody algebra.

The two-dimensional Virasoro $\times$ Virasoro algebra is represented both as a

- Symmetry of conformal field theories (CFTs)
- Subgroup of diffeomorphisms of gravitational theories which acts non-trivially

In some cases, the two representations are related by the AdS/CFT correspondence.

In the last years, the Virasoro $\times$ Kac-Moody algebra has appeared in various related contexts as

- the asymptotic symmetry algebra of warped AdS_3 spacetimes [G.C., Detournay, 2008]
- Generalizations of the Kerr/CFT correspondence [Guica, Hartman, Song, Strominger, 2008]
- an alternative symmetry enhancement of unitary QFTs with $SL(2,\mathbb{R}) \times U(1)$ symmetry [Hofman, Strominger, 2011]
- a sufficient symmetry to derive the entropy of warped AdS_3 black holes [Detournay, Hartman, Hofman, 2012]

However, no clear example of field theory or gravitational theory was proposed where warped symmetry is realized.

Given the universality of CFTs and their importance in holography, theories with warped conformal symmetries, if they exist, are expected to also play an important role.

In this talk, I will present a simple and tractable example where the warped symmetry is realized : chiral Liouville gravity.

The theory turns out to be consistent at the classical and semi-classical level.

The existence of a quantum theory is left for further work.

I will also show that this theory is dual to pure Einstein gravity in AdS_3 with appropriate boundary conditions.

In that sense, gravity in AdS_3 is not always dual to a CFT. It can also be dual to a warped CFT, depending on the boundary conditions.

Outline

Polyakov gravity and Liouville theory

The Polyakov action for $1+1$ quantum gravity is

$$S_P = \frac{c}{96\pi} \int d^2x \sqrt{-g}(R \,\Box^{-1} R - 2\Lambda)$$

or, more precisely,

$$S_P = \frac{c}{96\pi} \left(\int d^2x \int d^2x' \mathcal{R}(x) G(x,x') \mathcal{R}(x') - 2\Lambda \int d^2x \sqrt{-g} \right),$$

where

$$\begin{aligned} \mathcal{R} &\equiv \sqrt{-g} R, \\ \sqrt{-g} g^{ab} \nabla_a \nabla_b G(x,x') &= \delta^2(x,x'). \end{aligned}$$

The stress tensor defined by the metric variations of this action is

$$T_{ab} \equiv \frac{2\pi}{\sqrt{-g}} \frac{\delta S_P}{\delta g^{ab}}.$$

The trace of the stress tensor is the local expression

$$T^a{}_a = \frac{c}{24}(R + \Lambda).$$

In order to make sense of the theory, one usually gauge fixes part of the diffeomorphisms.

This is very subtle because diffeomorphisms are all pure gauge. They might act non-trivially.

Upon fixing the metric to light-cone “gauge”

$$g_{--} = 0 = g_{++},$$
$$-\frac{1}{2}e^{2\rho} \equiv g_{+-},$$

the action reduces to the local Liouville action

$$S_P = \frac{c}{12\pi}\int dt^+dt^-\left(\partial_-\rho\partial_+\rho - \frac{\Lambda}{8}e^{2\rho}\right).$$

Conformal symmetries preserves the light-cone gauge. They act non-trivially on ρ. Therefore, Liouville theory is a CFT.

The theory has been quantized. The theory is integrable.

Chiral Liouville gravity

We consider the chiral gauge fixing,

$$g_{--} = 0, \qquad \partial_-(g^{+-}g_{++}) = 0.$$

The metric is

$$ds^2 = -e^{2\rho}(dt^+dt^- - h(dt^+)^2).$$

$$\partial_- h = 0.$$

For $h \neq 0$, the gauge fixing is invariant under the warped conformal symmetries

$$\delta t^+ = \epsilon(t^+), \qquad \delta t^- = \sigma(t^+).$$

The diffeomorphisms $\delta t^- = \epsilon^-(t^-) \neq cst$ are not allowed.

We consider the Polyakov action with an additional chiral term,

$$\frac{c}{96\pi}\int d^2x\sqrt{-g}(R\,\square^{-1}R-2\Lambda)+\frac{\Delta}{4\pi}\int d^2x\sqrt{-g}g^{--}.$$

The chiral boundary term is invariant under warped conformal symmetries but not under standard conformal symmetries.

Using the equation of motion for g^{--}, it ensures that the left-moving stress-tensor zero mode is a fixed constant

$$T_{--}+\frac{\Delta}{2}=0.$$

This defines chiral Liouville theory at the classical level.

Chiral Liouville action

Using the gauge fixing condition $g_{--} = 0$, the action reduces to

$$S = \frac{c}{12\pi}\int dt^+ dt^- \Big(\partial_+\rho\partial_-\rho - \frac{\Lambda}{8}e^{2\rho} + h(\partial_-\rho)^2 - \frac{6}{c}h\Delta + \partial_- h\partial_-\rho + \frac{1}{2}\int d^2x'\partial_-^2 h(x)G(x,x')\partial_-^2 h(x')\Big).$$

Using the gauge fixing condition $\partial_-(g^{+-}g_{++}) = \partial_- h = 0$, the action becomes local

$$S = \frac{c}{12\pi}\int dt^+ dt^- \Big(\partial_+\rho\partial_-\rho - \frac{\Lambda}{8}e^{2\rho} + h\left[(\partial_-\rho)^2 - \frac{6}{c}\Delta - \partial_-^2\rho\right]\Big).$$

One can define the quantum theory at the semi-classical level using canonical quantization.

Conserved currents

The Noether procedure leads to the currents

$$j^a_\epsilon = e^{2\rho} T^a{}_-, \qquad j^a_\sigma = e^{2\rho} T^a{}_+.$$

where the stress-tensor is

$$T_{ab} = \frac{2\pi}{\sqrt{-g}} \frac{\delta L_P}{\delta g^{ab}} + \Delta \begin{pmatrix} h^2 & -\frac{3}{2}h \\ -\frac{h}{2} & \frac{1}{2} \end{pmatrix}_{ab}.$$

The full T_{ab} is not symmetric because of the asymmetric chiral action.

After imposing $T_{--} = 0$, $j^+_\epsilon = j^+_\sigma = 0$ and we are left with two right-moving conserved currents

$$\partial_- j^-_\epsilon = 0 = \partial_- j^-_\sigma.$$

Warped conformal symmetry algebra

Using canonical quantization, one can compute Dirac brackets.

The current-current commutators are obtained as

$$\begin{aligned}
[j^-_\sigma(t^+), j^-_\sigma(s^+)] &= \pi k\, \partial_{t^+}\delta(t^+ - s^+), \\
[j^-_\epsilon(t^+), j^-_\sigma(s^+)] &= 2\pi j^-_\sigma(t^+)\partial_{t^+}\delta(t^+ - s^+) \\
[j^-_\epsilon(t^+), j^-_\epsilon(s^+)] &= 2\pi\partial_{t^+}\delta(t^+ - s^-)(j^-_\epsilon(t^+) + j^-_\epsilon(s^+)) \\
&\quad -\frac{\pi c}{6}\partial^3_{t^+}\delta(t^+ - s^+).
\end{aligned}$$

This is a Virasoro Kac-Moody algebra with central charge c and level

$$k = -4\Delta.$$

Chiral Liouville gravity is integrable

The Bäcklund transformation is

$$\begin{aligned} e^{2\rho} &= -\frac{8}{\Lambda}\sqrt{\frac{6\Delta}{c}}e^{2O}\Big(\cosh\big(\sqrt{\frac{6\Delta}{c}}(t^- - P)\big) \\ &\quad - \sinh\big(\sqrt{\frac{6\Delta}{c}}(t^- - P)\big)\int dt^+ e^{2O}\Big)^{-2}, \\ h &= \partial_+ P, \end{aligned}$$

The equations of motion $R = -\Lambda$ and $T_{--} = 0$ then reduce to the free field equations

$$\partial_- O = 0, \quad \partial_- P = 0.$$

Evaluating the currents reveals

$$\begin{aligned} j^-_\epsilon &= \frac{c}{6}\Big((\partial_+ O)^2 - \partial_+^2 O\Big) - \Delta(\partial_+ P)^2, \\ j^-_\sigma &= 2\Delta\delta^{-+}\partial_+ P. \end{aligned}$$

Warped CFT as a deformed CFT

The chiral Liouville action is

$$S = \frac{c}{12\pi} \int dt^+ dt^- \left(\partial_+ \rho \partial_- \rho - \frac{\Lambda}{8} e^{2\rho} + h \left[(\partial_- \rho)^2 - \frac{6}{c} \Delta - \partial_-^2 \rho \right] \right).$$

The first two terms define the Liouville CFT.

Since $h = j_\sigma^-$, it has conformal weight $(1, 0)$. $\partial_-^2 \rho$ and $(\partial_- \rho)^2$ have conformal weight $(0, 2)$.

Therefore, the CFT is deformed by a $(1, 2)$ operator. This is connected to so-called dipole deformations.

This connection has not yet been understood.

Coupling to Matter

Take any QFT, e.g. a scalar field

$$\int d^2x \left(\partial_\mu \Phi \partial^\mu \Phi + m^2 \Phi^2\right).$$

One can define a new warped CFT by

- Making the action generally covariant and adding the Polyakov action
- Fixing to chiral gauge and fixing $T^{matter}_{--} = \frac{\Delta}{2}$

For the scalar field one obtains

$$S = S_P + \int dt^+ dt^- (\partial_+ \Phi \partial_- \Phi + h(\partial_- \Phi)^2 - \frac{m^2}{4} e^{2\rho} \Phi^2).$$

The resulting theory is a warped CFT.

Summary

- The Polyakov action in chiral gauge leads to Chiral Liouville Gravity
- The theory forms a non-trivial representation of the warped conformal symmetry

$$\delta t^+ = \epsilon(t^+), \qquad \delta t^- = \sigma(t^+).$$

  and is therefore a warped CFT.
- The theory can be defined semi-classically.
- The theory is integrable.
- An infinite number of warped CFTs can be built from coupling Chiral Liouville Gravity to any matter QFT.

Claim :

$1+1$ Chiral Liouville gravity

is holographically equivalent to AdS_3 Einstein gravity

with new chiral boundary conditions,

at the semi-classical level.

Milestones of AdS_3 Einstein gravity

1986, Brown-Henneaux Consistent boundary conditions with Virasoro $\times$ Virasoro asymptotic symmetries.

1993, Banados, Teitelboim, Zanelli AdS_3 Black holes

1995, Coussaert, Henneaux, van Driel AdS_3 Chern-Simons gravity with Brown-Henneaux boundary conditions is equivalent to Liouville theory

1998, Strominger BTZ entropy counting from dual CFT

2007, Maloney, Witten No consistent quantum AdS_3 Einstein gravity

The role of Brown-Henneaux boundary conditions is crucial.

Now, another set of boundary conditions exist.

Derivation of boundary conditions

The most general solution of Einstein gravity can be written as

$$ds^2 = \frac{dr^2}{r^2} + r^2(g^{(0)}_{ab} + \frac{1}{r^2} g^{(2)}_{ab} + \dots)$$

where

$$D^a_{(0)} g^{(2)}_{ab} \sim \partial_b R_{(0)}, \qquad g^{ab}_{(0)} g^{(2)}_{ab} \sim R_{(0)}.$$

Assuming no conformal anomaly $R_{(0)} = 0$ the variation of the Einstein action including covariant counterterms is

$$\delta S \sim \int d^2x \, g^{ab}_{(2)} \delta g^{(0)}_{ab}.$$

Known boundary conditions

Consistent boundary conditions only require

$$\delta S \sim \int d^2x \, g^{ab}_{(2)} \delta g^{(0)}_{ab} = 0.$$

Boundary conditions known before 2013 :

- Dirichlet [Brown-Henneaux] $g^{(0)}_{ab} = \bar{\eta}_{ab}$.
- Neumann [G.C., Marolf] $g^{ab}_{(2)} = 0$.

New boundary conditions :

- Chiral boundary conditions [G.C., Song, Strominger]

$$g^{(0)}_{--} = 0, \qquad g^{(0)}_{+-} = -\frac{1}{2}, \qquad g^{++}_{(2)} = \frac{16G}{\ell}\Delta.$$

- Generalized Dirichlet [Troessaert]

$$g^{(0)}_{ab} = e^{2\phi}\bar{g}_{ab}, \qquad \delta\bar{g}_{ab} = 0.$$

Chiral Boundary conditions

The boundary conditions can be written as

$$g^{(0)}_{--} = 0, \qquad g^{(0)}_{+-} = -\frac{1}{2}, \qquad g^{(0)}_{++} = \partial_+ \bar{P}(t^+),$$
$$g^{++}_{(2)} = \frac{16G}{\ell}\Delta.$$

The total action is defined as the Einstein action with Gibbons-Hawking and covariant counterterms S_0 complemented by a chiral boundary term,

$$S = S_0 + \frac{\Delta}{4\pi}\int dt^+ dt^- \sqrt{-g_{(0)}}\, g^{--}_{(0)}.$$

Phase space

The most general solution with chiral boundary condition contains the BTZ black holes with $\ell M = \Delta + \bar{\Delta}$ and $J = \Delta - \bar{\Delta}$

$$\frac{ds^2}{\ell^2} = \frac{dr^2}{r^2} - r^2 dt^+ dt^- + \frac{\bar{\Delta}}{k}(dt^+)^2 + \frac{\Delta}{k}(dt^-)^2 - \frac{\Delta\bar{\Delta}}{k^2 r^2} dt^+ dt^-,$$

dressed by boundary gravitons ($\bar{L}(t^+)$) and boundary photons ($\bar{P}(t^+)$). Here $k = \ell/4G$. It can be written as

$$\begin{aligned} ds^2 \quad &= \frac{\ell^2}{r^2} dr^2 - \ell^2 r^2 dt^+ \left(dt^- - \partial_+ \bar{P}(t^+) dt^+\right) \\ &+ 4G\ell \left[\bar{L}(t^+)(dt^+)^2 + \Delta\left(dt^- - \partial_+ \bar{P}(t^+) dt^+\right)^2\right] \\ &- \frac{16G^2\Delta}{r^2} \bar{L}(t^+) dt^+ \left(dt^- - \partial_+ \bar{P}(t^+) dt^+\right). \end{aligned}$$

Its form is preserved by the Kac-Moody-Virasoro asymptotic transformations

$$\delta t^+ = \epsilon(t^+), \qquad \delta t^- = \sigma(t^+),$$

which generate boundary gravitons $\bar{L}(t^+)$ and photons $\bar{P}(t^+)$. The associated charges are

$$\begin{aligned} Q_\epsilon &= \frac{1}{2\pi}\int_0^{2\pi} d\phi\, \epsilon(t^+)\left(\bar{L}(t^+) - \Delta(\partial_+\bar{P}(t^+))^2\right), \\ Q_\sigma &= \frac{1}{2\pi}\int_0^{2\pi} d\phi\, \sigma(t^+)(\Delta + 2\Delta\partial_+\bar{P}(t^+)). \end{aligned}$$

The Virasoro charge contains a Sugawara term.

Warped conformal asymptotic symmetry

Expanding in modes e^{int^+} we obtain the Dirac brackets

$$
\begin{aligned}
i\{\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n\} &= (m-n)\bar{\mathcal{L}}_{m+n} + \frac{c_R}{12} m^3 \delta_{m,-n}, \\
i\{\bar{\mathcal{L}}_m, \bar{\mathcal{J}}_n\} &= -n\bar{\mathcal{J}}_{m+n}, \\
i\{\bar{\mathcal{J}}_m, \bar{\mathcal{J}}_n\} &= \frac{k_{KM}}{2} m \delta_{m,-n}.
\end{aligned}
$$

The Virasoro central charge and level of the $U(1)$ Kac-Moody algebra are given by

$$
c_R = \frac{3\ell}{2G}, \qquad k_{KM} = -4\Delta\,.
$$

- AdS_3 ground state well-defined.
- Superradiance effect around black holes.

Lagrangian reduction

In order to dimensionally reduce AdS_3 Einstein gravity as a boundary theory, we switch to the Chern-Simons formulation,

$$S_E[A,\bar{A}] = S_k[A] + S_{-k}[\bar{A}] + (\text{Boundary terms})$$

where

$$S_k[A] \quad = \quad \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

and $k = \frac{\ell}{4G}$.

Boundary terms are fixed by the boundary conditions.

The chiral boundary conditions read as

$$
\begin{aligned}
A &= \begin{pmatrix} \frac{dr}{2r} & \frac{\Delta}{kr}(dt^- - \partial_+ \bar{P}(t^+)dt^+) \\ r(dt^- - \partial_+ \bar{P}(t^+)dt^+) & -\frac{dr}{2r} \end{pmatrix} \\
&\qquad + \begin{pmatrix} O(r^{-2}) & O(r^{-2}) \\ O(r^{-1}) & O(r^{-2}) \end{pmatrix}, \\
\bar{A} &= \begin{pmatrix} -\frac{dr}{2r} + O(r^{-2}) & rdt^+ + O(r^{-2}) \\ \frac{1}{kr}\bar{L}(t^+)dt^+ + O(r^{-2}) & \frac{dr}{2r} + O(r^{-2}) \end{pmatrix}.
\end{aligned}
$$

The chiral boundary conditions differ from the Brown-Henneaux ones only in the left sector.

We will only discuss the left sector. The right sector gives a chiral right-moving boson.

The equations $F = 0$ can be solved consistently with the boundary conditions by

$$A = G_L^{-1} dG_L, \qquad G_L = g_L(t^+, t^-) \begin{pmatrix} \sqrt{r} & 0 \\ 0 & \frac{1}{\sqrt{r}} \end{pmatrix},$$

where the Gauss decomposition gives

$$g_L = \begin{pmatrix} 1 & 0 \\ Y_L & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{-\Phi_L}{2}} & 0 \\ 0 & e^{\frac{\Phi_L}{2}} \end{pmatrix} \begin{pmatrix} 1 & X_L \\ 0 & 1 \end{pmatrix}.$$

The action is equivalent to

$$\begin{aligned} S_L &= \frac{k}{8\pi} \int_\Sigma dt^+ dt^- [-(\partial_+ \Phi_L \partial_- \Phi_L + 4e^{-\Phi_L} \partial_+ X_L \partial_- Y_L) \\ &\quad + 8 \frac{\Delta}{k} \frac{\partial_+ X_L}{\frac{\Delta}{k} - X_L^2}]. \end{aligned}$$

The boundary conditions imply further constraints among the fields X_L, Y_L, Φ_L :

$$
\begin{aligned}
\partial_+ X_L &= (X_L^2 - \frac{\Delta}{k})\partial_+ \bar{P}, \\
\partial_- X_L &= \frac{\Delta}{k} - X_L^2, \\
\partial_- \Phi_L &= -2X_L, \\
\partial_+ \Phi_L &= 2\partial_+ \bar{P} X_L, \\
\partial_+ Y_L &= -e^{\Phi_L}\partial_+ \bar{P}, \\
\partial_- Y_L &= e^{\Phi_L}.
\end{aligned}
$$

The general solution is two chiral bosons determined by $\bar{P}$ and Y_L. The right chiral boson is determined by X_R.

Combination of left and right sectors

We define the combined fields

$$e^{\Phi} = \frac{\partial_- Y_L \partial_+ X_R}{(1 - Y_L X_R)^2}, \qquad h = \partial_+ \bar{P}$$

The constraints are equivalent to the equations of motion of the chiral Liouville action

$$\begin{aligned} S &= \frac{k}{8\pi} \int dt^+ dt^- \left(\partial_+ \Phi \partial_- \Phi + 4e^{\Phi} + h((\partial_- \Phi)^2 - 2\partial_-^2 \Phi - \frac{4\Delta}{k}) \right), \\ \partial_- h &= 0. \end{aligned}$$

Moreover, the Dirac brackets of the Virasoro-Kac-Moody generators are equivalent.

Therefore, the Lagrangian reduction of AdS_3 gravity is equivalent to chiral Liouville gravity.

Conclusion

- We obtained the first $1+1$ Warped Conformal Field Theory with Virasoro $\times$ $U(1)$ Kac-Moody symmetry.
- Chiral Liouville theory can be obtained from Polyakov gravity in chiral gauge.
- It can also be obtained holographically from AdS_3 Einstein gravity with chiral boundary conditions.
- The theory is integrable and defined semi-classically.
- The theory can be coupled to matter, as ordinary Liouville theory can.

Important question left : what about the quantum theory ?

Other question : what does it tell us about black holes ?