The $\mathcal{N} = 2$ Gauss-Bonnet invariant in and out of superspace

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Based on work with B. de Wit, S. Kuzenko, and I. Lodato

Motivation

The Gauss-Bonnet is a very nice higher derivative invariant:

$$L_{\rm GB} = \frac{1}{4}\varepsilon^{mnpq}R_{mn}{}^{ab}R_{pq}{}^{cd}\varepsilon_{abcd} = C^{abcd}C_{abcd} - 2\mathcal{R}^{ab}\mathcal{R}_{ab} + \frac{2}{3}\mathcal{R}^2 \ .$$

Although it is topological by itself, it often appears multiplied by a scalar function in specific applications (e.g. anomalies, 5D to 4D reductions, etc.). Its supersymmetric version will appear in corresponding situations.

The manifestly supersymmetric 4D $\mathcal{N}=1$ GB is well-known.
[Townsend and van Nieuwenhuizen; Ferrara and Villasante; Buchbinder and Kuzenko]
This reflects the completeness of our understanding of 4D $\mathcal{N}=1$ higher derivative terms.

We should better understand higher derivative terms in 4D $\mathcal{N}=2$!

Outline

1. Superspace philosophy
2. The $\mathcal{N} = 1$ Gauss-Bonnet in superspace and an $\mathcal{N} = 2$ mystery
3. Superconformal tensor calculus in superspace
4. The Gauss-Bonnet invariant in and out of $\mathcal{N} = 2$ superspace

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Quick review: $\mathcal{N} = 1$ supersymmetry in spinor notation

In a Weyl-basis, the γ-matrices are

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad \gamma^m = \begin{pmatrix} 0 & (\sigma^m)_{\alpha\dot\alpha} \\ (\bar\sigma^m)^{\dot\alpha\alpha} & 0 \end{pmatrix}$$

A Dirac fermion Ψ and its conjugate $\bar\Psi$ look like

$$\Psi = \begin{pmatrix} \chi_\alpha \\ \bar\psi^{\dot\alpha} \end{pmatrix} , \qquad \bar\Psi = \begin{pmatrix} \psi^\alpha & \bar\chi_{\dot\alpha} \end{pmatrix}$$

$\mathcal{N} = 1$ supersymmetry in four dimensions:

$$\{Q_\alpha, \bar Q_{\dot\alpha}\} = -2i\,(\sigma^m)_{\alpha\dot\alpha}\partial_m \ .$$

Superspace makes supersymmetry manifest

Superspace: add new Grassmann coordinates θ_α and $\bar\theta^{\dot\alpha}$

Interpret the supersymmetry generator $Q_\alpha \sim D_\alpha$ as

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i(\sigma^m)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_m$$

Chiral multiplet is "short"

$$\bar D_{\dot\alpha}\Phi = 0 \qquad \Longrightarrow \qquad \Phi = \phi + \theta^\alpha\psi_\alpha + \theta^2 F + (x\text{-derivative terms})$$

The free massless chiral multiplet action is given by

$$\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\,\bar\Phi\Phi = \int \mathrm{d}^4x\left(\bar\phi\Box\phi - \frac{i}{2}\bar\psi_{\dot\alpha}(\bar\sigma^m)^{\dot\alpha\alpha}\partial_m\psi_\alpha + F\bar F\right)$$

If we want to gauge a $U(1)$ symmetry, the standard lore is to add an explicit vector multiplet prepotential V, do Wess-Zumino gauge, etc.

How to avoid prepotentials and Wess-Zumino gauges

Encode gauge connection in superspace: $A = \mathrm{d}x^m A_m \to \mathcal{A} = \mathrm{d}z^M \mathcal{A}_M$.

Build covariant derivative $\mathcal{D}_A = (\mathcal{D}_\alpha, \bar{\mathcal{D}}^{\dot{\alpha}}, \mathcal{D}_a)$ in superspace.
This is supersymmetric minimal substitution.

Keep the same action $\bar{\Phi}\Phi$ but replace the flat chiral constraint with a covariant chiral constraint $\bar{D}^{\dot{\alpha}}\Phi = 0 \longrightarrow \bar{\mathcal{D}}^{\dot{\alpha}}\Phi = 0$.

We define the components of Φ *not* by a θ expansion but by

$$\phi \equiv \Phi|_{\theta=0} \;, \qquad \psi_\alpha = \mathcal{D}_\alpha \Phi|_{\theta=0} \;, \qquad F = -\frac{1}{4}\mathcal{D}^\alpha \mathcal{D}_\alpha \Phi|_{\theta=0}.$$

The supergravity story is essentially analogous.

1. Lift all connections to superconnections.
2. Replace flat derivatives with covariant derivatives.

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From superspace to tensor calculus and back again

For every covariant field e.g. ψ_α, there is some superfield, e.g. $\mathcal{D}_\alpha \Phi$ with $\psi_\alpha = \mathcal{D}_\alpha \Phi|_{\theta=0}$ Supersymmetry on the component field corresponds to covariant spinor derivative on the superfield:

$$\delta_Q \psi_\alpha = \xi^\beta Q_\beta \psi_\alpha + \bar{\xi}_{\dot\beta} \bar{Q}^{\dot\beta} \psi_\alpha$$

$$\delta_\xi \mathcal{D}_\alpha \Phi|_{\theta=0} = \xi^\beta \mathcal{D}_\beta \mathcal{D}_\alpha \Phi|_{\theta=0} + \bar{\xi}_{\dot\beta} \bar{\mathcal{D}}^{\dot\beta} \mathcal{D}_\alpha \Phi|_{\theta=0}$$

Given covariant superfields, one can evaluate all their components and derive supersymmetry transformations. The converse is also possible: one can often *lift* component results to superspace expressions.

Properties/symmetries of the tensor calculus reflect those of the superspace.

$$\text{Poincaré tensor calculus} \Longleftrightarrow \text{Poincaré superspace}$$

$$\text{Superconformal tensor calculus} \Longleftrightarrow \text{(Super)conformal superspace}$$

Outline

$\mathcal{N} = 1$ Poincaré superspace

Old minimal Poincaré supergravity involves the field content

$$e_m{}^a, \quad \psi_m{}^\alpha, \quad \underbrace{V_m}_{\text{massive vector}}, \quad \underbrace{M}_{\text{complex scalar}}$$

Superspace geometry involves curvatures which are built out of the superfields $W_{\alpha\beta\gamma}$, G_a , R. These contain (respectively) C_{abcd}, $\mathcal{R}_{ab}$ and $\mathcal{R}$.

General integrals in superspace look like

$$S_D = \int \mathrm{d}^4x \, e \, [\mathcal{L}]_D = \int \mathrm{d}^4x \, \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \, E \, \mathcal{L} \, , \quad \mathcal{L} \text{ is unconstrained}$$

$$S_F = \int \mathrm{d}^4x \, e \, [\mathcal{L}_c]_F = \int \mathrm{d}^4x \, \mathrm{d}^2\theta \, \mathcal{E} \, \mathcal{L}_c \, , \quad \bar{\mathcal{D}}^{\dot{\alpha}} \mathcal{L}_c = 0$$

A caveat: any D-term can be written as an F-term.
Let's consider only "honest" F-terms.

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$\mathcal{N} = 1$ actions and higher derivatives

The Einstein-Hilbert Lagrangian is

$$L_{\rm EH} = -3M_P^2 \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar\theta\, E \sim -\frac{1}{2} M_P^2 \int \mathrm{d}^4x\, e\, \mathcal{R}$$

In $\mathcal{N} = 1$, the only purely chiral (non-singular) invariants are

$$\mathcal{L}_c = c_1 W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} + \frac{c_2}{M_P^3} (W^{\alpha\beta\gamma} W_{\alpha\beta\gamma})^2 + \cdots$$

$$[\mathcal{L}_c]_F \sim c_1 C^{-abcd} C^-_{abcd} + \frac{c_2}{M_P^3} (C^-)^2 (\mathcal{D}\psi)^2 + \cdots$$

The other (non-singular) terms generically are D-term invariants with

$$\mathcal{L} = c_3 R\bar{R} + c_4 G^a G_a + c_5 \mathcal{D}^\alpha \mathcal{D}_\alpha R + \frac{c_6}{M_P^4} (\mathcal{D}_\alpha G_a)^4 + \frac{c_7}{M_P^3} (W_{\alpha\beta\gamma})^2 G^2 + \cdots$$

$$[\mathcal{L}_D] \sim c_3 \mathcal{R}^2 + c_4 \mathcal{R}^{ab} \mathcal{R}_{ab} + c_5 \Box \mathcal{R} + \frac{c_6}{M_P^4} (\mathcal{R}_{ab})^4 + \frac{c_7}{M_P^3} (C^-)^2 (\mathcal{D}\psi)^2 + \cdots$$

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The Gauss-Bonnet invariant is a certain combination of the highlighted terms. Moreover, the full supersymmetric result (with all the fermions) is *topological*.

The mystery in $\mathcal{N} = 2$

For simplicity, let's stick with the minimal $\mathrm{SU}(2)$ multiplet:

$$e_m{}^a\,, \quad \psi_{m\alpha}{}^i\,, \quad \mathcal{V}_m{}^i{}_j\,, \quad W_m\,, \qquad \Longleftarrow \text{gauge connections}$$
$$T_{ab}{}^{ij}\,, \quad \chi_\alpha^i\,, \quad D\,, \quad Y_{ij}\,, \quad V_m \qquad \Longleftarrow \text{covariant fields}$$

Superspace geometry includes curvatures

$$\underbrace{W_{\alpha\beta}}_{\text{chiral Weyl superfield}}\,, \quad S^{ij}\,, \quad G_a\,, \quad Y_{\alpha\beta}$$

The minimal multiplet action is the F-term integral of a constant

$$S_{\text{minimal}} = -\frac{3}{2}M_P^2 \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathcal{E} = M_P^2 \int \mathrm{d}^4x\, e \Big(-\frac{1}{2}\mathcal{R} + 3D + \cdots\Big)$$

The obvious higher derivative chiral invariant is

$$\mathcal{L}_c = W^{\alpha\beta} W_{\alpha\beta} \quad \Longleftrightarrow \quad [\mathcal{L}_c]_F \sim (C^-)^2$$

D-term invariants lead to dimension 6 and higher:

$$\mathcal{L} = \frac{1}{M_P^2}\Big(S^{ij}S_{ij} + Y^{\alpha\beta}Y_{\alpha\beta} + \cdots\Big)\,, \qquad [\mathcal{L}]_D \sim \frac{1}{M_P^2}\Big(\mathcal{R}^3 + \mathcal{R}\Box\mathcal{R} + \cdots\Big)$$

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Review: Conformal gravity

Let's briefly review conformal gravity since the superconformal case is quite similar.

- Introduce covariant derivative

$$e_m{}^a \nabla_a = \partial_m - \frac{1}{2}\omega_m{}^{ab} M_{ab} - b_m \mathbb{D} - f_m{}^a K_a$$

  so that $\Box_c = \nabla^a \nabla_a$.

- We may consistently impose curvature constraints to determine $\omega_m{}^{ab}$ and $f_m{}^a$. Only independent fields are $e_m{}^a$ and b_m.

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$\mathcal{N} = 2$ superconformal vs. Poincaré tensor calculus

For $\mathcal{N} = 2$ conformal supergravity, we have the fundamental connections

$$e_m{}^a\ , \quad \psi_m{}^{\alpha i}\ , \quad b_m\ , \quad A_m\ , \quad \mathcal{V}_m{}^i{}_j$$

Constraints $\implies$ composite $\omega_m{}^{ab}$, $f_m{}^a$ and $\phi_m{}^\alpha{}_i$

Choosing a certain compensator, we can reduce to a Poincaré tensor calculus

- real scalar multiplet (128+128)
  $\mathrm{U}(2)$ Poincaré tensor calculus $\sim$ $\mathrm{U}(2)$ superspace [Howe '81]
- chiral multiplet (16+16)
  $\mathrm{SU}(2)$ Poincaré tensor calculus $\sim$ $\mathrm{SU}(2)$ superspace [Grimm '80]
- vector multiplet (8+8)
  minimal $\mathrm{SU}(2)$ Poincaré tensor calculus $\sim$ reduced $\mathrm{SU}(2)$ superspace

Poincaré supergravity multiplets still can describe superconformal theories. Then the additional compensator fields drop out of the action. But this can be complicated.

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$$e_m{}^a\,, \quad \psi_m{}^{\alpha i}\,, \quad b_m\,, \quad A_m\,, \quad \mathcal{V}_m{}^i{}_j$$

Constraints $\implies$ composite $\omega_m{}^{ab}$, $f_m{}^a$ and $\phi_m{}^\alpha{}_i$

Choosing a certain compensator, we can reduce to a Poincaré tensor calculus

- real scalar multiplet (128+128)
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Poincaré supergravity multiplets still can describe superconformal theories. Then the additional compensator fields drop out of the action. But this can be complicated.

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$\mathcal{N} = 2$ conformal superspace

For a superconformal model, use conformal superspace (no compensator!)

[DB '11]

- Introduce covariant derivative $\nabla_A = (\nabla_{\alpha i}, \bar{\nabla}^{\dot{\alpha} i}, \nabla_a)$ with connections in the full superconformal algebra.
- Single curvature superfield $W_{\alpha\beta}$ contains all of the superconformal curvatures and the "matter" fields $T_{ab}{}^{ij}\ , \chi_\alpha^i\ , D$
- All constraints and curvatures of $\mathcal{N} = 2$ superconformal tensor calculus are reproduced.
- This was worked out long ago at the linearized level in superspace.

[Bergshoeff, de Roo, de Wit '81]

Outline

Gauss-Bonnet invariant in conformal gravity

- $\Box_c\Box_c \ln\phi$ is K-invariant for any choice of weight w for ϕ and equals

$$\Box\Box\ln\phi + \mathcal{D}^a\Big(\frac{2}{3}\mathcal{R}\mathcal{D}_a\ln\phi - 2\mathcal{R}_{ab}\mathcal{D}^b\ln\phi\Big) + \frac{w}{6}\Box\mathcal{R} - \frac{w}{2}\mathcal{R}^{ab}\mathcal{R}_{ab} + \frac{w}{6}\mathcal{R}^2\ .$$

- We can get very close to the usual Gauss-Bonnet in conformal gravity:

$$\begin{aligned} L'_{\rm GB} &= C^{abcd}C_{abcd} + \frac{4}{w}\Box_c\Box_c\ln\phi \\ &= L_{\rm GB} + \frac{2}{3}\Box\mathcal{R} + \frac{4}{w}\mathcal{D}^a\Big(\mathcal{D}_a\Box\ln\phi + \tfrac{2}{3}\mathcal{R}\mathcal{D}_a\ln\phi - 2\mathcal{R}_{ab}\mathcal{D}^b\ln\phi\Big) \end{aligned}$$

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This combination is known to physicists as the Riegert operator.

[Riegert '84; Paneitz '08]

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- This allows you to see that under a finite Weyl transformation

$$\begin{aligned} \sqrt{g}\Big(L_{\rm GB} + \frac{2}{3}\Box\mathcal{R}\Big) \Longrightarrow \ & \sqrt{g}\Big(L_{\rm GB} + \frac{2}{3}\Box\mathcal{R}\Big) \\ & - 4\sqrt{g}\mathcal{D}^a\Big(\mathcal{D}_a\Box\sigma + \frac{2}{3}\mathcal{R}\mathcal{D}_a\sigma - 2\mathcal{R}_{ab}\mathcal{D}^b\sigma\Big) \end{aligned}$$

Supersymmetrizing the Gauss-Bonnet invariant

Let us take the point of view that we wish to supersymmetrize

$$L'_{\rm GB} = C^{abcd} C_{abcd} + \frac{4}{w} \Box_{\rm c} \Box_{\rm c} \ln \phi$$

This can be done using superconformal methods.

- The first term should supersymmetrize easily to the superconformal Weyl-squared invariant.
- The second term can be supersymmetrized once we decide on the multiplet for ϕ. For both $\mathcal{N} = 1, 2$, the natural choice is to take ϕ to be complex and the lowest component of a chiral multiplet.

For both $\mathcal{N} = 1, 2$ the natural quantity will be complex:

$$L_\Gamma = \frac{1}{8} (C^-)^2 + \frac{1}{4w} \Box_{\rm c} \Box_{\rm c} \ln \bar\phi + \cdots$$

$\mathcal{N} = 1$ conformal superspace construction

In $\mathcal{N} = 1$ conformal superspace, only a single superspace curvature $W_{\alpha\beta\gamma}$.

[DB '09]

The only option is an F-term action with $\mathcal{L}_c = W^{\alpha\beta\gamma} W_{\alpha\beta\gamma}$

$$[W^{\alpha\beta\gamma} W_{\alpha\beta\gamma}]_F = \frac{1}{16} C_{abcd} C^{abcd} - \frac{1}{16} C_{abcd} \tilde{C}^{abcd} + \text{additional terms} .$$

In flat space, it is easy to check that $-\frac{1}{64} \int \mathrm{d}^2\theta \, \bar{D}^2 D^2 \bar{D}^2 \ln \overline{\Phi} = \Box\Box \ln \bar{\phi}$. Can we covariantize?

Take chiral superfield Φ of weight w and introduce

$$\mathbb{S}(\ln \overline{\Phi}) = -\frac{1}{64} \bar{\nabla}^2 \nabla^2 \bar{\nabla}^2 \ln \overline{\Phi} .$$

One can check that $\mathbb{S}$ is chiral and also S-invariant. This means we can use it to build chiral actions.

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We choose the chiral F-term Lagrangian is

$$\Gamma := W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} + \frac{1}{4w} \mathbb{S}(\ln \overline{\Phi}) \ ,$$

$$[\Gamma]_F = \frac{1}{8}(C^-)^2 + \frac{1}{4w} \Box_c \Box_c \ln \bar{\phi} + \text{additional terms}$$

Rewriting in $\mathcal{N} = 1$ Poincaré superspace...

$$\mathbb{S}(\ln \overline{\Phi}) = -\frac{1}{4}(\bar{\mathcal{D}}^2 - 8R)\Big(\frac{1}{16}\mathcal{D}^2 \bar{\mathcal{D}}^2 \ln \overline{\Phi} + \mathcal{D}^\alpha (G_{\alpha\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}} \ln \overline{\Phi}) + 4wG^a G_a + 8wR\bar{R} - \frac{1}{2}w\mathcal{D}^2 R\Big)$$

We can see that we recover the topological $\mathcal{N} = 1$ Gauss-Bonnet combination

$$\int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathcal{E}\, \Gamma = \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathcal{E}\, W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} + \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\, E \left(2R\bar{R} + G^a G_a - \frac{1}{8}\mathcal{D}^2 R\right)$$

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$\mathcal{N} = 2$ conformal superspace construction

Again: we want to supersymmetrize

$$L_\Gamma = \frac{1}{8}(C^-)^2 + \frac{1}{4w}\Box_c\Box_c \ln\bar\phi$$

The first term is easy. The second resembles the $N = 2$ kinetic multiplet.

[de Wit, Katmadas, van Zalk '11]

In flat $\mathcal{N} = 2$ superspace, the second term is generated by

$$\int d^4\theta\, \bar D^4 \ln\overline{\Phi} = D^4\bar D^4 \ln\overline{\Phi}|_{\theta=0} = \Box\Box\ln\bar\phi$$

In $\mathcal{N} = 2$ conformal superspace, we can simply covariantize: $\mathbb{T} := \bar\nabla^4 \ln\overline{\Phi}$. We call this the *nonlinear kinetic multiplet*. For this to be a valid chiral Lagrangian, we must check that it is chiral and S-invariant,

$$\bar\nabla^{\dot\alpha i}\mathbb{T} = \bar S_i^{\dot\alpha}\mathbb{T} = S_\alpha^i\mathbb{T} = 0\ .$$

Then $\mathbb{T}$ is a proper conformally primary chiral multiplet, and we can write

$$\int d^4x\, d^4\theta\,\mathcal{E}\,\mathbb{T} = \int d^4x\, e\,\Box_c\Box_c\ln\bar\phi + \cdots$$

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But how do we actually evaluate the last equation?

Some details of chiral multiplets in superspace

In $\mathcal{N} = 2$ superspace, we need to know how to convert F-terms to component actions.

$$\int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathcal{E}\, \mathcal{L}_c = \int \mathrm{d}^4x\, e \left(\nabla^4 \mathcal{L}_c + \cdots\right)|_{\theta=0}$$

The exact formula coincides with the usual chiral invariant density of $\mathcal{N} = 2$ superconformal tensor calculus:

$$\begin{aligned}[\mathcal{L}_c]_F \propto C &- \varepsilon^{ij}\, \bar\psi_{\mu i}\gamma^\mu \Lambda_j - \tfrac{1}{8}\bar\psi_{\mu i} T_{ab\,jk}\gamma^{ab}\gamma^\mu \Psi_l\, \varepsilon^{ij}\varepsilon^{kl} - \tfrac{1}{16} A (T_{ab\,ij}\varepsilon^{ij})^2 \\ &- \tfrac{1}{2}\bar\psi_{\mu i}\gamma^{\mu\nu}\psi_{\nu j}\, B_{kl}\, \varepsilon^{ik}\varepsilon^{jl} + \varepsilon^{ij}\bar\psi_{\mu i}\psi_{\nu j}(F^{-\mu\nu} - \tfrac{1}{2} A\, T^{\mu\nu}{}_{kl}\, \varepsilon^{kl}) \\ &- \tfrac{1}{2}\varepsilon^{ij}\varepsilon^{kl} e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar\psi_{\mu i}\psi_{\nu j}(\bar\psi_{\rho k}\gamma_\sigma \Psi_l + \bar\psi_{\rho k}\psi_{\sigma j}\, A)\end{aligned}$$

where $A, \cdots, C$ are the components of the chiral multiplet $\mathcal{L}_c$.

In our case, we need to know the chiral components corresponding to $\mathbb{T} := \bar\nabla^4 \ln \overline{\Phi}$.

Components of the nonlinear kinetic multiplet

The components of the $\ln \overline{\Phi}$ multiplet

$$\hat{\bar{A}} := \ln \overline{\Phi}|_{\theta=0} , \quad \hat{\bar{\Psi}}^{\dot{\alpha} i} := \bar{\nabla}^{\dot{\alpha} i} \ln \overline{\Phi}|_{\theta=0} , \quad \cdots , \quad \hat{\bar{C}} :\propto \bar{\nabla}^4 \ln \overline{\Phi}$$

The components of the nonlinear kinetic multiplet $\mathbb{T} := \bar{\nabla}^4 \ln \overline{\Phi}$

$$A|_{\mathbb{T}} = \mathbb{T}|_{\theta=0} , \quad \Psi_{\alpha i}|_{\mathbb{T}} = \nabla_{\alpha i} \mathbb{T}|_{\theta=0} , \quad \cdots , \quad C|_{\mathbb{T}} \propto \nabla^4 \mathbb{T}|_{\theta=0}$$

A straightforward (increasingly tedious) calculation, using $\nabla_{\alpha i} \ln \overline{\Phi} = 0$, gives

$$\begin{aligned} &A|_{\mathbb{T}} \propto \hat{\bar{C}} , \quad \cdots \quad , \\ &C|_{\mathbb{T}} \propto \nabla^4 \bar{\nabla}^4 \ln \overline{\Phi}|_{\theta=0} = \Box_{\mathrm{c}} \Box_{\mathrm{c}} \ln \bar{\phi} + \cdots \end{aligned}$$

After the dust settles...

We have a new invariant

$$
\begin{aligned}
[\mathbb{T}]_F \propto\; & 4(\Box_{\rm c} + 3\,D)\Box_{\rm c} \log\bar\phi - \tfrac{1}{2} D_a\big(T^{ab}{}_{ij}\,T_{cb}{}^{ij}\big)\,D^c \log\bar\phi \\
& + D_a\big(\varepsilon^{ij} D^a T_{bcij}\,\hat F^{+bc} + 4\,\varepsilon^{ij} T^{ab}{}_{ij}\,D^c \hat F^+_{cb} - T_{bc}{}^{ij}\,T^{ac}{}_{ij}\,D^b\log\bar\phi\big) \\
& + \big(6\,D_b D - 8{\rm i} D^a R(A)_{ab}\big)\,D^b\log\bar\phi \\
& - wR(\mathcal{V})^{+}_{ab}{}^{i}{}_{j} R(\mathcal{V})^{+\,abj}{}_{i} + 8w\,R(D)^+_{ab} R(D)^{+\,ab} \\
& - wD^c\big(D_a T^{ab\,ij} T_{cb\,ij}\big) - wD_a T^{ab\,ij} D^c T_{cb\,ij} + \cdots\,,
\end{aligned}
$$

The Weyl squared invariant is

$$
\begin{aligned}
L_{\rm W^2} \propto\; & C^{abcd}C_{abcd} - C^{abcd}\tilde C_{abcd} - 4R(A)^{-ab} R(A)^-_{ab} + R(\mathcal{V})^{-abi}{}_j R(\mathcal{V})^-_{ab}{}^j{}_i \\
& + 6D^2 - T^{ac\,ij} D_a D^b T_{bc\,ij} - \frac{1}{128} T^{ab\,ij} T_{ab}{}^{kl} T^{cd}{}_{ij} T_{cd\,kl}
\end{aligned}
$$

Putting them together in the right combination we find (up to an explicit total derivative)

$$
\begin{aligned}
& C^{abcd}C_{abcd} - 2\mathcal{R}^{ab}\mathcal{R}_{ab} + \frac{2}{3}\mathcal{R}^2 - C^{abcd}\tilde C_{abcd} \\
& + 2R(A)^{ab}\tilde R(A)_{ab} - R(\mathcal{V})_{ab}{}^i{}_j \tilde R(\mathcal{V})^{abj}{}_i
\end{aligned}
$$

So we have a new invariant, and it turns out to match the combination we needed to find from dimensional reduction from 5D... But what about the mystery of the minimal multiplet?

A new chiral invariant in the minimal multiplet!

Let's try to understand the nonlinear kinetic multiplet by writing it in $SU(2)$ superspace:

$$\bar\nabla^4 \ln \overline{\Phi} = \Delta \ln \overline{\Phi} + w\mathbb{T}_0$$
$$\mathbb{T}_0 = \frac{1}{12}\bar{\mathcal{D}}_{ij}\bar{S}^{ij} + \frac{1}{2}\bar{S}_{ij}\bar{S}^{ij} + \frac{1}{2}\bar{Y}_{\dot\alpha\dot\beta}\bar{Y}^{\dot\alpha\dot\beta} \ .$$

The operator Δ is the $SU(2)$ superspace chiral operator, generalizing $\bar{D}^4$ of flat superspace. Under a full superspace integral, one can show that

$$\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{E}\,\Delta \ln \overline{\Phi} = \int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathrm{d}^4\bar\theta\,E\,\ln \overline{\Phi} = 0$$

so the dependence on $\overline{\Phi}$ lies only within a total derivative.
The remaining combination $\mathbb{T}_0$ must be chiral (this can be checked explicitly), which is remarkable since none of its individual pieces is chiral!

The minimal multiplet has an additional chiral invariant!

Conclusions / Open questions (1/2)

We have constructed a new chiral invariant based on $\mathcal{N} = 2$ conformal supergravity coupled to a chiral multiplet.

- If the chiral multiplet is taken to be a vector multiplet and gauged to unity, it gives a new chiral invariant in the $\mathrm{SU}(2)$ minimal multiplet.
- It corresponds to certain actions which arise from reduction from 5D.
- The chiral multiplet can also be considered composite. Then in addition to D-term invariants

$$\int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathrm{d}^4\bar\theta\, E\, \mathcal{H}(X, \bar X)\,, \qquad X^I \mathcal{H}_I = 0$$

we have intrinsic chiral invariants that look like

$$\int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathcal{E}\, \Phi'(X) \mathbb{T}(\ln \bar\Phi(\bar X))\,, \qquad X^I \Phi'_I = 0\,, \qquad \bar X^{\bar J} \bar\Phi_{\bar J} = w \bar\Phi$$

In flat space, the second class corresponds to the first with the choice $\mathcal{H} = \Phi' \ln \bar\Phi$, but not in the curved case. In fact, a term of this form was exactly seen from dimensional reduction from 5D.

Open questions (2/2)

But there are several things we don't know.

- Does the new invariant class contribute to black hole entropy?

  but hopefully soon...
- The new invariant is peculiar in that it is (almost) independent of the compensator. Can we construct generic $\mathcal{R}^2$ and $(\mathcal{R}_{ab})^2$ terms by introducing the second (tensor?) comensator? In principle, this should be so.